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A double integration method for generating exact tolerance limit factors for normal populations

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Abstract. This article introduces a new method for generating the exact one-sided and two-sided tolerance limit factors for normal populations. This method does not need to handle the noncentral *t*-distribution at all, but only needs to do a double integration of a joint probability density function with respect to the two independent variables "s" (standard deviation) and "$\overline{x}$" (sample mean). The factors generated by this method are investigated through Monte Carlo simulations and compared with the existing factors. As a result, it is identified that the two-sided tolerance limit factors being currently used in practical applications are inaccurate. For the right understanding, some factors generated by this method are presented in Tables along with a guidance for correct use of them. The AQL (Acceptable Quality Level) is a good, common measure about quality of a product lot which was already produced or will be produced. Therefore, when performing sampling inspection on a given lot using a tolerance limit factor, there is a necessity to know the AQL assigned to the factor. This new double integration method even makes it possible to generate the AQLs corresponding to the one-sided and two-sided tolerance limit factors.

Keywords: AQL / acceptable quality level / sampling inspection / sample size / tolerance limit factor / tolerance interval

1 Introduction

The one-sided and two-sided tolerance limit factors are needed for quality control by sampling inspection. In response to this necessity, for the past tens of years, a lot of methodologies for generating the factors have been developed. Wald and Wolfowitz [1] developed an approximate formula for setting the tolerance limits in 1946. Even since then, other approximate formulas were developed. Representatively, Lieberman [2] developed a formula for one-sided tolerance limit factors and it was cited widely in other popular references (e.g., Natrella [3]). Unfortunately, Lieberman's formula tends to underestimate the factors; as the sample size n decreases, the factor gets much more underestimated. For this reason, it could be used only for sample sizes larger than 50. In 1970, Guttman [4] developed an approximate formula for one-sided tolerance limit factors using the statistical properties of the noncentral *t*-distribution. For reference, tens of years ago, it was not easy to use directly the noncentral *t*-distribution. Johnson and Welch [5] encouraged applications of the noncentral *t*-distribution through their research results. Owen and Amos [6] suggested some computational programs needed to use the noncentral *t*-distribution in order to set the tolerance limits. In 1972, Abramowitz and Stegun [7] proved that the noncentral *t*-distribution can be approximated by the normal distribution. With the aid of Abramowitz and Stegun's theory, Link [8] also suggested an approximate formula, and then Link compared the three formulas for one-sided tolerance limit factors (i.e., Lieberman [2], Guttman [4], and Link [8]). In comparatively recent years, Janiga and Garaj [9], and Young [10] presented their respective computational methodologies. In this way, various research activities relating to the one-sided tolerance limit factors have been done for a long time. It seems that most of recent studies are focused mainly on improvement or simplification of computational procedures by statistical software rather than exact calculation of the factors.

Although related studies are still ongoing, fortunately, the sufficiently accurate one-sided tolerance limit factors which were generated by Owen [11] in 1963 are being currently used in industry. Owen created each factor by an iterative method based on the noncentral *t*-distribution with degrees of freedom ν and noncentrality parameter δ. Later in 1993, Wheeler [12] reviewed and compared two methods for generating tolerance limit factors which were

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called "Method A" and "Method B" in his paper. "Method A" was a method based on the noncentral t-distribution, and "Method B" was a unique method which does not need to utilize the noncentral t-distribution. The calculation results by the two methods are the same. The one-sided tolerance limit factors generated by Wheeler coincide with the ones by Owen [11]. From this, in the case of one-sided, Wheeler's two methods are all correct and exact as well as Owen's method.

In parallel with the efforts to develop the methodologies for generating the one-sided tolerance limit factors, a lot of effort to generate exactly the two-sided tolerance limit factors has also been made till now. In 1964, Owen [13] used the term "equal-tailed tolerance intervals" for the first time and developed methodologies relating to the two-sided tolerance limit factors. Howe [14] also improved relevant methodologies. Odeh et al. [15] wrote a pocket book containing the diverse statistical constants needed for computing normal tolerance intervals. Odeh [16], and Odeh and Owen [17] created tables of the two-sided tolerance limit factors. In 1993, Wheeler [12] tried to calculate the two-sided tolerance limit factors using the two methods (i.e., "Method A" and "Method B") in the same manner as used in the case of one-sided. In comparatively recent years, Garaj and Janiga [18], Jensen [19], Krishnamoorthy and Mathew [20], Krishnamoorthy and Xie [21], Witkovsky [22], and Young [23] presented their respective methodologies or computational algorithms for calculating the two-sided tolerance limit factors.

Unfortunately, despite such efforts, in the case of two-sided, the exact factors have not been established yet. Although both of Wheeler's two methods [12] can exactly generate the one-sided tolerance limit factors, either of the two methods does not exactly generate the two-sided tolerance limit factors. It is difficult to reflect correctly the concept of the two-sided tolerance limit factors in the calculation of the factors. In addition, the calculation procedures are more complicated than one-sided. These are presumably the reason that any correct methodology has not been established yet. Nevertheless, the two-sided tolerance limit factors generated inaccurately are being used in industry without any criticism or comment and it is a problem.

In this article, a new method for generating correctly the one-sided and two-sided tolerance limit factors is introduced as the solution for the problem. This method does not need to handle the noncentral t-distribution at all, but only needs to utilize the statistical properties of normal distribution and chi-square distribution, which makes the calculation process much easier. To generate the factors, this method needs a double integration of a joint probability density function with respect to the two independent variables "s" (standard deviation) and "$\overline{x}$" (sample mean). The related definite integrals are evaluated numerically by a statistical software. Notably, after a one-sided or two-sided tolerance limit factor has been generated, the AQL (Acceptable Quality Level) corresponding to that factor can also be derived by the same double integration method. For demonstration purposes, some one-sided and two-sided tolerance limit factors, including some AQLs, obtained by the new method are presented in tables. Through Monte Carlo simulations, it is identified that this new method is correct and exact and that the existing two-sided tolerance limit factors which were generated by other methods are inaccurate. In addition, a guidance for correct use of the tolerance limit factors is presented.

2 Method and determination of factors

2.1 Definitions of k_1, k_2, $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$

The four tolerance limit factors k_1, k_2, $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$ are defined. (The practical tolerance limit factors $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$ are determined by the factors k_1 and k_2 in a proper manner.) First, imagine that an inspector needs to perform a sampling inspection on a given normal population according to the inspection parameters of sample size n, confidence level $(1-\alpha)$ and proportion p. Here, $0<p<1$ and $0<\alpha<1$.

Let's assume that not "at least", but "exactly" the proportion p of the given normal population lies below the upper specification limit U (or above the lower specification limit L) and let's assume that the inspector tests only this population. The factor k_1 is defined as the one-sided tolerance limit factor which will be used by the inspector to ensure with confidence level $(1-\alpha)$ that the proportion p of the normal population is below U (or above L). When the inspector performs the one-sided test using the factor k_1, the probability that the population does not pass the test is "exactly $(1-\alpha)$".

Let's assume that not "at least", but "exactly" the proportion p of the given normal population lies between L and U, and the population mean is equal to $(U+L)/2$. In addition, let's assume that the inspector tests only this population. The factor k_2 is defined as the two-sided tolerance limit factor which will be used by the inspector to ensure with confidence level $(1-\alpha)$ that the proportion p of the normal population lies between L and U. When the inspector performs the two-sided test using the factor k_2, the probability that the population does not pass the test is "exactly $(1-\alpha)$".

The factor $k_{1,\text{practical}}$ is defined as the one-sided tolerance limit factor which can be directly used in all practical applications to ensure with confidence level $(1-\alpha)$ that at least the proportion p of any given normal population is below U (or above L). In case that more than the proportion p of the normal population is below U (or above L), although k_1 is used instead of $k_{1,\text{practical}}$ as the one-sided tolerance limit factor, the inspector can conservatively ensure with confidence level $(1-\alpha)$ that at least the proportion p of the normal population is below U (or above L). Therefore, k_1 can be selected as substitute for $k_{1,\text{practical}}$ in all practical applications.

The factor $k_{2,\text{practical}}$ is defined as the two-sided tolerance limit factor which can be directly used in all practical applications to ensure with confidence level $(1-\alpha)$ that at least the proportion p of any given normal population lies between L and U. Here, unlike the case of k_1 and $k_{1,\text{practical}}$, k_2 cannot always be selected as substitute for $k_{2,\text{practical}}$. Whether or not k_2 can be used instead of $k_{2,\text{practical}}$ as the two-sided tolerance limit factor in a practical application depends on the situation. More specifically, if k_2 is larger than its corresponding factor k_1, k_2 can be selected as substitute for

$k_{2,\text{practical}}$. However, if k_2 is smaller than k_1, k_2 cannot be selected as substitute for $k_{2,\text{practical}}$. Instead k_1 should be selected; the detailed reason is explained in Section 3.

Throughout this article, when referring to the term "two corresponding factors k_1 and k_2", it includes the meaning that the inspection parameters (i.e., sample size, confidence level, and proportion) given to the factor k_1 are identical to those given to the factor k_2. Similarly, when referring to the term "AQL corresponding to factor k_1 (or k_2)", it includes the meaning that the inspection parameters given to the AQL are identical to those given to the factor k_1 (or k_2). These remarks are also effective for $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$. For example, in this Section 2.1, the factors k_1, k_2, $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$ have been defined based on the same inspection parameters (i.e., the same sample size n, the same confidence level $(1-\alpha)$, and the same proportion p). Therefore, in this case, it can be said that the four factors correspond to one another.

2.2 Properties of *s*-distribution

A distribution named *s*-distribution is defined. Let's assume that a normal population of the random variable x is given and n samples (i.e., sample size is n) are randomly selected from it. If the same sampling process is repeated numerously, we can envision the probability distribution of the standard deviation s $(s=\{\sum_{i=1}^{n}(x_i-\overline{x})^2/(n-1)\}^{1/2}$ and $\overline{x}=\sum_{i=1}^{n}x_i/n)$. Here, such probability distribution is defined as *s*-distribution. To proceed further, some basic information is reminded. If x is the normal random variable and its variance is σ^2, then the sample mean $\overline{x}$ is normally distributed independently of s and the statistic $(n-1)s^2/\sigma^2$ follows the chi-square distribution with degrees of freedom ν $(=n-1)$. Based on such definition of *s*-distribution, we can proceed as follows.

Let $s'=\{(n-1)^{1/2}/\sigma\}s=as$, $a=(n-1)^{1/2}/\sigma$, and $h=s'^2$. In addition, let $\varphi(s)$ be the probability density function of the continuous variable s, let $\psi(s')$ be the probability density function of s', and let $\phi(h)$ be the probability density function of $h(=s'^2)$. Then the following can be found.

$$\begin{aligned}\int_0^t\varphi(s)ds &= \int_0^{at}\psi(s')ds'\left(\text{let } s'=as,\quad \text{then } \frac{ds'}{ds}=a\right)\\ &=\int_0^t\psi(as)a\,ds\end{aligned}.$$

This equality holds for all real numbers t $(t>0)$.

$$\therefore \varphi(s)=a\psi(as)=a\psi(s'). \tag{1}$$

Similarly,

$$\begin{aligned}\int_0^t\psi(s')ds' &= \int_0^{t^2}\phi(h)dh\left(\text{let } h=s'^2,\quad \text{then } \frac{dh}{ds'}=2s'\right)\\ &=\int_0^t\phi(s'^2)2s'ds'\end{aligned},$$

$$\therefore \psi(s')=2s'\phi\left(s'^2\right). \tag{2}$$

2.3 Determination of k_1 by double integration

According to the definition of k_1 in Section 2.1, let's assume that not "at least", but "exactly" the proportion p of the given normal population lies below the upper specification limit U and let's assume that an inspector tests only this population. In addition, suppose that the inspector needs to ensure with confidence level $(1-\alpha)$ by a sampling inspection that the proportion p of the normal population is below U, where $0<p<1$ and $0<\alpha<1$. If the mean and the variance of the population are μ and σ^2 respectively and the sample size is n, then the one-sided tolerance limit factor k_1 to be used for the sampling inspection should satisfy the following equation.

$$\int_0^{\infty}\varphi(s)\left\{\int_{-\infty}^{(\mu+K_p\sigma)-k_1s}\Phi(\overline{x})d\overline{x}\right\}ds=\alpha. \tag{3}$$

The factor K_p is the critical value corresponding to the proportion p in one-tailed test of normal population, and $\Phi(\overline{x})$ is the probability density function of the normal variable $\overline{x}$ with the mean μ and the variance σ^2/n. Using equations (1) and (2), equation (3) can be rewritten in the following form.

$$\begin{aligned}\alpha &= \int_0^{\infty}a\psi(as)\left\{\int_{-\infty}^{(\mu+K_p\sigma)-k_1s}\Phi(\overline{x})d\overline{x}\right\}ds\\ &=\int_0^{\infty}a\psi(s')\left\{\int_{-\infty}^{(\mu+K_p\sigma)-k_1s'/a}\Phi(\overline{x})d\overline{x}\right\}\left(\frac{1}{a}\right)ds'\\ &=\int_0^{\infty}2s'\phi\left(s'^2\right)\left\{\int_{-\infty}^{(\mu+K_p\sigma)-k_1s'/a}\Phi(\overline{x})d\overline{x}\right\}ds'\end{aligned}. \tag{4}$$

It should be reminded that $\phi(s'^2)$ is the probability density function of the variable s'^2 with degrees of freedom ν $(=n-1)$. Equation (4) does not need to handle the noncentral *t*-distribution. Only the probability density functions of normal distribution and chi-square distribution are needed to solve it. However, this equation can be solved numerically, not analytically.

In real situation, to determine numerically the factor k_1, the probability density function in equation (4) should be integrated with respect to s'. In principle, its integration range should be from 0 to ∞. However, the range of 0 to 10 (or the range of 0 to 20 for large sample size, i.e., n > 40) is sufficient, because $\phi(s'^2)$ and/or the definite integral of $\Phi(\overline{x})$ is nearly 0 for s' greater than 10 (or 20) and hence, does not affect the integration results. The definite integral with respect to s' is evaluated numerically by "division quadrature" with the aid of a statistical software (e.g., program "MINITAB 15"). Each unit interval of s' from 0 to 10 (or 20) is divided into 100,000 equal parts for the division quadrature.

As an example of determination of k_1, let $\mu=500$ and $\sigma^2=5^2$. (Although other μ and σ^2 are selected, the same result is obtained.) In addition, let $p=0.95$, $(1-\alpha)=0.95$,

Table 1. Numerically determined factor $\boldsymbol{k}_1$.

Sample size n	$1-\alpha=0.75$ $p=0.75$	$1-\alpha=0.90$ $p=0.90$	$1-\alpha=0.90$ $p=0.95$	$1-\alpha=0.95$ $p=0.90$	$1-\alpha=0.95$ $p=0.95$	$1-\alpha=0.99$ $p=0.99$
2	2.225	10.252	13.089	20.580	26.258	185.543
3	1.464	4.258	5.311	6.155	7.656	23.895
4	1.255	3.188	3.957	4.162	5.144	12.387
5	1.152	2.742	3.400	3.407	4.203	8.939
6	1.088	2.494	3.092	3.006	3.708	7.335
7	1.043	2.333	2.894	2.755	3.400	6.412
8	1.010	2.219	2.754	2.582	3.187	5.812
9	0.985	2.133	2.650	2.454	3.031	5.389
10	0.964	2.066	2.568	2.355	2.911	5.074
11	0.947	2.011	2.503	2.275	2.815	4.829
12	0.932	1.966	2.448	2.210	2.736	4.633
13	0.920	1.928	2.402	2.155	2.671	4.472
14	0.909	1.895	2.363	2.109	2.614	4.337
15	0.899	1.867	2.329	2.068	2.566	4.222
16	0.891	1.842	2.299	2.033	2.524	4.123
17	0.883	1.820	2.272	2.002	2.486	4.037
18	0.876	1.800	2.249	1.974	2.453	3.960
19	0.870	1.782	2.227	1.949	2.423	3.893
20	0.864	1.765	2.208	1.926	2.396	3.832

and $n=15$, then equation (4) is rewritten to determine its corresponding k_1 as follows.

$$0.05=\int_0^{\infty or 10} 2s'\phi\left(s'^2\right)\{\int_{-\infty}^{(500+1.64485x5)-k_1 s'/0.74833}\Phi(\overline{x})d\overline{x}\}ds',$$

$K_p=K_{0.95}=1.64485$ and $a=(15-1)^{1/2}/5=0.74833$ have been substituted into equation (4). Both by "division quadrature" and by "trial and error", we find that k_1 is 2.566. (It will take a few minutes for an expert accustomed to this process to generate a factor needed.) In this way, all k_1 factors can be numerically determined. Table 1 shows some k_1 factors thus obtained. For reference, the one-sided tolerance limit factors being currently used in industry (Owen [11], Wheeler [12] and ISO [24]) coincide with the k_1 factors which are determined by the above method.

2.4 Determination of k_2 by double integration

According to the definition of k_2 in Section 2.1, let's assume that not "at least", but "exactly" the proportion p of the given normal population lies between the lower specification limit L and the upper specification limit U, and the population mean is equal to $(U+L)/2$. In addition, let's assume that an inspector tests only this population. Also suppose that the inspector needs to ensure with confidence level $(1-\alpha)$ by a sampling inspection that the proportion p of the normal population lies between L and U, where $0<p<1$ and $0<\alpha<1$. If the mean and the variance of the normal population are μ and σ^2 respectively, and the sample size is n, then the two-sided tolerance limit factor k_2 to be used for the sampling inspection should satisfy the following equation.

$$\begin{aligned}\alpha &= \int_0^{K'_p\sigma/k_2}\varphi(s)\{\int_{\mu}^{(\mu+K'_p\sigma)-k_2 s}\Phi(\overline{x})d\overline{x}\}ds \\ &+\int_0^{K'_p\sigma/k_2}\varphi(s)\{\int_{k_2 s+(\mu-K'_p\sigma)}^{\mu}\Phi(\overline{x})d\overline{x}\}ds \qquad (5)\\ &=2\int_0^{K'_p\sigma/k_2}\varphi(s)\{\int_{\mu}^{(\mu+K'_p\sigma)-k_2 s}\Phi(\overline{x})d\overline{x}\}ds \qquad (6)\end{aligned}$$

The factor K'_p is the critical value corresponding to the proportion p in two-tailed test of normal population. On the right-hand side of equation (5), the first term includes the definite integral with respect to the normal variable $\overline{x}$ greater than μ, and the second term includes the definite integral with respect to $\overline{x}$ smaller than μ. In the same manner as used in the case of one-sided, equation (6) is changed into the integral expression with respect to the variable s' as follows

$$\alpha=2\int_0^{aK'_p\sigma/k_2} 2s'\phi\left(s'^2\right)\{\int_{\mu}^{(\mu+K'_p\sigma)-k_2 s'/a}\Phi(\overline{x})d\overline{x}\}ds'. \qquad (7)$$

Table 2. Numerically determined factor $\boldsymbol{k_2}$.

Sample size n	$1-\alpha=0.75$ $p=0.75$	$1-\alpha=0.90$ $p=0.90$	$1-\alpha=0.90$ $p=0.95$	$1-\alpha=0.95$ $p=0.90$	$1-\alpha=0.95$ $p=0.95$	$1-\alpha=0.99$ $p=0.99$
2	1.905[a]	8.629[a]	11.109[a]	17.304[a]	22.272[a]	160.441[a]
3	1.385[a]	3.779[a]	4.722[a]	5.433[a]	6.782[a]	21.377[a]
4	1.270	2.937[a]	3.629[a]	3.801[a]	4.691[a]	11.317[a]
5	1.223	2.597[a]	3.191[a]	3.192[a]	3.917[a]	8.291[a]
6	1.199	2.413[a]	2.953[a]	2.875[a]	3.514[a]	6.884[a]
7	1.184	2.297[a]	2.804[a]	2.680[a]	3.266[a]	6.075[a]
8	1.175	2.217[a]	2.700[a]	2.547[a]	3.098[a]	5.551[a]
9	1.168	2.157	2.623[a]	2.449[a]	2.975[a]	5.182[a]
10	1.163	2.112	2.564[a]	2.375	2.881[a]	4.907[a]
11	1.159	2.075	2.517	2.316	2.806[a]	4.695[a]
12	1.157	2.045	2.479	2.268	2.746	4.525[a]
13	1.154	2.020	2.446	2.228	2.695	4.386[a]
14	1.153	1.999	2.419	2.194	2.653	4.269[a]
15	1.151	1.981	2.395	2.165	2.616	4.170[a]
16	1.150	1.965	2.375	2.140	2.584	4.085[a]
17	1.149	1.950	2.356	2.118	2.556	4.011[a]
18	1.148	1.938	2.340	2.098	2.531	3.945[a]
19	1.148	1.927	2.326	2.080	2.509	3.887[a]
20	1.147	1.916	2.312	2.064	2.488	3.835

[a] Each of these two-sided tolerance limit factors is smaller than its corresponding one-sided tolerance limit factor. For example, if $n=5$, $(1-\alpha)=0.99$, and $p=0.99$, then k_2 is 8.291 and k_1 is 8.939 (see Table 1).

Only the probability density functions of normal distribution and chi-square distribution are needed to solve equation (7). However, this equation cannot be solved analytically, but can be solved numerically like equation (4).

Similar to the case of k_1, let's determine numerically a factor k_2 by equation (7). First, let $\mu=500$ and $\sigma^2=5^2$, and let $p=0.95$, $(1-\alpha)=0.95$, and $n=15$. Then equation (7) is rewritten in the following form to determine its corresponding k_2.

See equation below.

Both by "division quadrature" and by "trial and error", we obtain $k_2=2.616$. At this time, $K'_p=K'_{0.95}=1.95996$ has been used as the critical value corresponding to the proportion 0.95 in two-tailed test of normal population. In this way, all k_2 factors can be numerically determined. Table 2 shows some k_2 factors thus obtained. From Tables 1 and 2, when the sample size n is small, some k_2 factors are smaller than their respective corresponding k_1 factors. As the proportion p and the confidence level $(1-\alpha)$ come closer to 1, this phenomenon becomes more apparent. For reference, the two-sided tolerance limit factors being currently used in industry (Wheeler [12] and ISO [24]) are different from the k_2 factors which are determined by the above method.

2.5 Simulation study

The factors k_1 and k_2 presented in Tables 1 and 2 can be investigated by Monte Carlo simulations. As an example, one two-sided tolerance limit factor is investigated to identify its correctness. In the case of $n=10$, $p=0.90$, and $(1-\alpha)=0.90$, the two-sided tolerance limit factor k_2 numerically determined is 2.112 from Table 2. Before simulating with this factor, suppose that the mean μ and the variance σ^2 of the normal population to be two-sided tested are 500 and 5^2 respectively. After then, 491.77575 $(=\mu-K'_p\sigma=500–1.64485\times 5)$ and 508.22425 $(=\mu+K'_p\sigma=500+1.64485\times 5)$ are assigned as the lower specification limit L and the upper specification limit U respectively so that exactly the proportion 0.90 of the population lies between L and U. As the first stage of the simulation, the sample mean $\overline{x}$ and the standard deviation s of the ten independent random numbers generated from the given normal population $(\mu=500,\ \sigma^2=5^2)$ by a statistical software (e.g., program "MINITAB 15") are calculated. Then it is observed whether or not the lower tolerance limit $(\overline{x}-k_2s)$ is larger than L, and whether or not the upper tolerance limit $(\overline{x}+k_2s)$ is smaller than U. This observation is repeated 1,000,000 times. Only in cases where $(\overline{x}-k_2s)$ is larger than L and simultaneously $(\overline{x}+k_2s)$ is smaller than U, the two-sided test confirms that the population is acceptable.

$$0.05 = 2\int_{0}^{0.74833\,\mathrm{x}1.95996\,\mathrm{x}\,5/k_2} 2s'\phi\left(s'^2\right)\{\int_{500}^{(500+1.95996\,\mathrm{x}\,5)-k_2s'/0.74833}\Phi(\overline{x})d\overline{x}\}ds'.$$

Table 3. Simulation result.

Sample size n	$1-\alpha=0.75,\ p=0.75$		$1-\alpha=0.90,\ p=0.90$		$1-\alpha=0.99,\ p=0.99$	
One–sided	k_1	Result $(1–\alpha)$[a]	k_1	Result $(1–\alpha)$[a]	k_1	Result $(1–\alpha)$[a]
2	2.225	0.750145	10.252	0.899861	185.543	0.990003
5	1.152	0.749197	2.742	0.899606	8.939	0.990014
10	0.964	0.750319	2.066	0.900001	5.074	0.989846
15	0.899	0.749729	1.867	0.900345	4.222	0.989946
20	0.864	0.749845	1.765	0.899290	3.832	0.990018
Two–sided	k_2	Result $(1–\alpha)$[a]	k_2	Result $(1–\alpha)$[a]	k_2	Result $(1–\alpha)$[a]
2	1.905	0.750599	8.629	0.899754	160.441	0.990025
5	1.223	0.749557	2.597	0.899523	8.291	0.989940
10	1.163	0.749948	2.112	0.900041	4.907	0.990035
15	1.151	0.750224	1.981	0.900113	4.170	0.990117
20	1.147	0.750703	1.916	0.900088	3.835	0.989854

[a] Each of the $(1-\alpha)$ values was estimated from 1,000,000 observations.

Table 4. Comparison of k_2 and $k_{2,\text{existing}}$.

k_2 determined by new integration method and $k_{2,\text{existing}}$ being currently used in industry

Sample size n	$1-\alpha=0.75,\ p=0.75$		$1-\alpha=0.90,\ p=0.90$		$1-\alpha=0.99,\ p=0.99$	
	k_2	$k_{2,\text{existing}}$	k_2	$k_{2,\text{existing}}$	k_2	$k_{2,\text{existing}}$
2	1.905	4.393[a]	8.629	15.512[a,b]	160.441	234.878[a,b]
5	1.223	1.829[a]	2.597	3.499[a,b]	8.291	10.220[a,b]
10	1.163	1.496[a]	2.112	2.546[a,b]	4.907	5.610[a,b]
15	1.151	1.398[a]	1.981	2.285[a,b]	4.170	4.621[b]
20	1.147	1.349[a]	1.916	2.158[a,b]	3.835	4.175[b]
25	1.145	1.319[a]	1.877	2.081[a]	3.638	N/A[c]
30	1.144	1.299[a]	1.851	2.029[a,b]	3.507	3.743[b]
35	1.143	1.284[a]	1.831	1.991[a,b]	3.412	3.619[b]
40	1.143	1.272[a]	1.816	1.961[a,b]	3.339	3.524[b]
50	1.143	1.255[a]	1.794	1.918[a,b]	3.235	3.390[b]
100	1.143	N/A[c]	1.744	1.823[b]	3.005	3.098[b]

[a] These factors are the ones that were provided by Wheeler [12].
[b] These factors are the ones that were provided by ISO [24].
[c] N/A means "not applicable". Either Wheeler or ISO did not provide the two-sided tolerance limit factors for some sampling inspection parameters.

In the simulation, only 99,959 observations showed that $(\overline{x} - k_2 s)$ was larger than L (= 491.77575) and simultaneously $(\overline{x} + k_2 s)$ was smaller than U (= 508.22425). From this, the observed α value is 99,959/1,000,000 = 0.099959; in other words, the confidence level identified by the simulation is $(1-\alpha) = 0.900041$. Therefore, it can be said that the two-sided tolerance limit factor $k_2 = 2.112$ determined numerically by the new integration method is correct. In this way, the correctness of all the one-sided and two-sided tolerance limit factors can be investigated by Monte Carlo simulations. Table 3 shows the results from the simulations that were conducted to investigate some k_1 and k_2 factors.

In Table 4, for $n = 10$, $p = 0.90$, and $(1–\alpha) = 0.90$, the two-sided tolerance limit factor being used in industry is 2.546. This factor is significantly different from the above factor 2.112; the factor 2.546 is the one that was unduly overestimated. Table 4 additionally shows that regardless of sample size, confidence level, and proportion, the $k_{2,\text{existing}}$ factors are always larger than the k_2 factors. Therefore, from Tables 3 and 4, it can be said that the two-sided tolerance limit factors being used in industry are inaccurate. However, we can observe that as the sample size n increases, the difference between k_2 and $k_{2,\text{existing}}$ gets smaller.

3 Practical factors and guidance for correct use

According to the definition of $k_{1,\text{practical}}$ in Section 2.1, even in case that more than the proportion p of any given normal population is below U (or above L), the use of $k_{1,\text{practical}}$ should enable inspectors to ensure with confidence level $(1-\alpha)$ that at least the proportion p of the normal population is below U (or above L). Fortunately, although its corresponding factor k_1 is used as the one-sided tolerance limit factor instead of $k_{1,\text{practical}}$, the inspectors can conservatively ensure the requirement with confidence level $(1-\alpha)$. Therefore, k_1 can be selected as substitute for $k_{1,\text{practical}}$ in all practical applications. Thus, we can say

$$k_{1,practical} = k_1(\text{in all cases}). \tag{8}$$

According to the definition of $k_{2,\text{practical}}$, even in case that more than the proportion p of any given normal population lies between L and U, and moreover, the population mean is not equal to $(U+L)/2$, the use of $k_{2,\text{practical}}$ should enable inspectors to ensure with confidence level $(1-\alpha)$ that at least the proportion p of the normal population lies between L and U. Here, unlike the case of k_1 and $k_{1,\text{practical}}$, k_2 cannot always be selected as substitute for $k_{2,\text{practical}}$. Whether or not k_2 can be used instead of $k_{2,\text{practical}}$ depends on the situation. Imagine the two corresponding factors k_1 and k_2 which were determined according to the same inspection parameters (i.e., the same sample size n, the same confidence level $(1-\alpha)$, and the same proportion p). Here, if k_2 is larger than k_1, k_2 can be selected as substitute for $k_{2,\text{practical}}$. On the contrary, if k_2 is smaller than k_1 (see the k_2 factors marked with superscript "a" in Table 2), k_2 cannot be selected as substitute for $k_{2,\text{practical}}$. Instead k_1 should be selected. To summarize,

$$k_{2,\text{practical}} = \begin{bmatrix} k_1 & (if\, k_2 < k_1) \\ k_2 & (if\, k_2 > k_1) \end{bmatrix}. \tag{9}$$

In case that the population mean μ is not equal to $(U+L)/2$, i.e., the population mean does not coincide with the center of the specification range, the factor $k_{2,\text{practical}}$ will be affected by the deviation magnitude of μ from the center. If the population mean comes closer to U (or L) and the variance gets smaller, the sampling inspection will become more similar to the one-sided test. Moreover, the inspector performs the test, not knowing the mean and variance of the given normal population. Therefore, if k_2 is not larger than its corresponding factor k_1, k_1 (not k_2) should be selected as substitute for $k_{2,\text{practical}}$ for the two-sided test. This is the reason that the determination of $k_{2,\text{practical}}$ depends on whether k_2 is larger than k_1 or not. For reference, the two-sided tolerance limit factors being currently used in industry (Wheeler [12] and ISO [24]) are always larger than their respective corresponding one-sided tolerance limit factors.

Although this assumption is not realistic, let's assume that the inspector knows the mean of a given normal population to be tested. In this case, $k_{2,\text{practical}}$ needed for the two-sided test can be directly and numerically determined, not relying on k_1 and k_2. As an example, suppose that the inspector needs to perform a two-sided test on the normal population according to the inspection parameters of $n=5$, $(1-\alpha)=0.95$, $p=0.95$, $L=490$, and $U=510$. In addition, let's assume that the population mean μ is known to be 503.4206. Then, $k_{2,\text{practical}}$ (here, denoted by k'_2 for convenience during calculation) can be determined by the following equation. (Before calculating, we assign 4^2 as the variance σ^2 of the population so that nearly the proportion $p=0.95$ of the population lies between L and U.) $(503.4206 = 510–1.64485 \times 4)$

$$\begin{aligned} \alpha &= \int_0^{a(U-\mu)/k_2'} 2s'\phi\left(s'^2\right)\{\int_\mu^{U-k_2's'/a}\Phi(\overline{x})d\overline{x}\}ds' \\ &+ \int_0^{a(U-\mu)/k_2'} 2s'\phi\left(s'^2\right)\{\int_{L+k_2's'/a}^{\mu}\Phi(\overline{x})d\overline{x}\}ds' \\ &+ \int_{a(U-\mu)/k_2'}^{a(U-L)/2k_2'} 2s'\phi\left(s'^2\right)\{\int_{L+k_2's'/a}^{U-k_2's'/a}\Phi(\overline{x})d\overline{x}\}ds' \\ &= \int_0^{a(U-\mu)/k_2'} 2s'\phi\left(s'^2\right)\{\int_\mu^{U-k_2's'/a}\Phi(\overline{x})d\overline{x}\}ds' \\ &+ \int_0^{a(U-L)/2k_2'} 2s'\phi\left(s'^2\right)\{\int_{L+k_2's'/a}^{\mu}\Phi(\overline{x})d\overline{x}\}ds' \\ &- \int_{a(U-\mu)/k_2'}^{a(U-L)/2k_2'} 2s'\phi\left(s'^2\right)\{\int_{U-k_2's'/a}^{\mu}\Phi(\overline{x})d\overline{x}\}ds' \end{aligned}. \tag{10}$$

Notably, in the last term on the right-hand side of equation (10), $U-k'_2s'/a$ is smaller than μ. If we substitute related parameters, i.e., $n=5$, $\alpha=0.05$, $L=490$, $U=510$, $\mu=503.4206$, $\sigma^2=4^2$ and $a=(n-1)^{1/2}/\sigma=0.5$, into equation (10) and then solve it using both "division quadrature" and "trial and error" in a similar manner to determination of k_1 and k_2, we obtain k'_2 $(= k_{2,\text{practical}}) = 4.156$, which is needed for the two-sided test. However, if n is 15, a different result is obtained. At this time, if we substitute $n=15$, $\alpha=0.05$, $L=490$, $U=510$, $\mu=503.4206$, $\sigma^2=4^2$ and $a=(n-1)^{1/2}/\sigma=0.93541$ into equation (10) and then solve it, we obtain $k'_2(= k_{2,\text{practical}}) \fallingdotseq 2.566$. The above two calculation results are summarized as follows.

when $n=5$; $k_2=3.917 < k'_2(= k_{2,\text{practical}})=4.156 < k_1=4.203$

when $n=15$; $k_1=2.566 \fallingdotseq k'_2(= k_{2,\text{practical}}) \fallingdotseq 2.566 < k_2=2.616$

Thus, irrespective of sample size n, the factor $k_{2,\text{practical}}$ calculated directly is always placed somewhere between the two corresponding factors k_1 and k_2. However, as previously mentioned, inspectors usually perform two-sided tests (including one-sided tests), not knowing the mean and variance of normal populations, and hence cannot calculate the $k_{2,\text{practical}}$ factors. Therefore, $k_{2,\text{practical}}$ should be selected from the two corresponding factors k_1 and k_2 by equation (9). Tables 5 and 6 show some $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$ factors that should be used in practical applications.

Table 5. Practical factor $k_{1,\text{practical}}$ $(= k_1)$.

Sample size n	$1-\alpha=0.75$ $p=0.75$	$1-\alpha=0.90$ $p=0.90$	$1-\alpha=0.90$ $p=0.95$	$1-\alpha=0.95$ $p=0.90$	$1-\alpha=0.95$ $p=0.95$	$1-\alpha=0.99$ $p=0.99$
2	2.225	10.252	13.089	20.580	26.258	185.543
3	1.464	4.258	5.311	6.155	7.656	23.895
4	1.255	3.188	3.957	4.162	5.144	12.387
5	1.152	2.742	3.400	3.407	4.203	8.939
6	1.088	2.494	3.092	3.006	3.708	7.335
7	1.043	2.333	2.894	2.755	3.400	6.412
8	1.010	2.219	2.754	2.582	3.187	5.812
9	0.985	2.133	2.650	2.454	3.031	5.389
10	0.964	2.066	2.568	2.355	2.911	5.074
11	0.947	2.011	2.503	2.275	2.815	4.829
12	0.932	1.966	2.448	2.210	2.736	4.633
13	0.920	1.928	2.402	2.155	2.671	4.472
14	0.909	1.895	2.363	2.109	2.614	4.337
15	0.899	1.867	2.329	2.068	2.566	4.222
16	0.891	1.842	2.299	2.033	2.524	4.123
17	0.883	1.820	2.272	2.002	2.486	4.037
18	0.876	1.800	2.249	1.974	2.453	3.960
19	0.870	1.782	2.227	1.949	2.423	3.893
20	0.864	1.765	2.208	1.926	2.396	3.832

Table 6. Practical factor $k_{2,\text{practical}}$.

Sample size n	$1-\alpha=0.75$ $p=0.75$	$1-\alpha=0.90$ $p=0.90$	$1-\alpha=0.90$ $p=0.95$	$1-\alpha=0.95$ $p=0.90$	$1-\alpha=0.95$ $p=0.95$	$1-\alpha=0.99$ $p=0.99$
2	2.225[a]	10.252[a]	13.089[a]	20.580[a]	26.258[a]	185.543[a]
3	1.464[a]	4.258[a]	5.311[a]	6.155[a]	7.656[a]	23.895[a]
4	1.270	3.188[a]	3.957[a]	4.162[a]	5.144[a]	12.387[a]
5	1.223	2.742[a]	3.400[a]	3.407[a]	4.203[a]	8.939[a]
6	1.199	2.494[a]	3.092[a]	3.006[a]	3.708[a]	7.335[a]
7	1.184	2.333[a]	2.894[a]	2.755[a]	3.400[a]	6.412[a]
8	1.175	2.219[a]	2.754[a]	2.582[a]	3.187[a]	5.812[a]
9	1.168	2.157	2.650[a]	2.454[a]	3.031[a]	5.389[a]
10	1.163	2.112	2.568[a]	2.375	2.911[a]	5.074[a]
11	1.159	2.075	2.517	2.316	2.815[a]	4.829[a]
12	1.157	2.045	2.479	2.268	2.746	4.633[a]
13	1.154	2.020	2.446	2.228	2.695	4.472[a]
14	1.153	1.999	2.419	2.194	2.653	4.337[a]
15	1.151	1.981	2.395	2.165	2.616	4.222[a]
16	1.150	1.965	2.375	2.140	2.584	4.123[a]
17	1.149	1.950	2.356	2.118	2.556	4.037[a]
18	1.148	1.938	2.340	2.098	2.531	3.960[a]
19	1.148	1.927	2.326	2.080	2.509	3.893[a]
20	1.147	1.916	2.312	2.064	2.488	3.835

[a] In case that k_2 is smaller than its corresponding factor k_1, k_1 (not k_2) was selected as substitute for $k_{2,\text{practical}}$ (see Tables 1 and 2).

4 AQLs corresponding to factors

According to Owen [11], AQL is defined as "percentage defective in a lot corresponding to a 0.95 chance of accepting the lot". The AQL is a good, common measure about quality of a product lot as a whole, which was already produced or will be produced. Therefore, when testing a given lot using a tolerance limit factor, there is a necessity to know the AQL which is assigned to the sampling inspection. In this regard, the method for generating the AQLs corresponding to the tolerance limit factors is introduced.

Suppose that the mean μ and the variance σ^2 of the normal population to be two-sided tested are 500 and 5^2 respectively and that the inspection parameters are $n = 5$, $(1-\alpha) = 0.90$, and $p = 0.90$. Also assume that the population

Table 7. AQLs corresponding to k_1 and k_2

One–sided						
Sample size n	$1-\alpha = 0.90,\ p = 0.90$		$1-\alpha = 0.90,\ p = 0.95$		$1-\alpha = 0.99,\ p = 0.99$	
	k_1	AQL	k_1	AQL	k_1	AQL
2	10.252	28.86%[a]	13.089	21.99%	185.543	nearly 0%
5	2.742	16.66%	3.400	10.05%	8.939	0.011%
10	2.066	13.81%	2.568	7.75%	5.074	0.131%
15	1.867	12.88%	2.329	7.04%	4.222	0.240%
20	1.765	12.40%	2.208	6.68%	3.832	0.318%
Two–sided						
Sample size n	$1-\alpha = 0.90,\ p = 0.90$		$1-\alpha = 0.90,\ p = 0.95$		$1-\alpha = 0.99,\ p = 0.99$	
	k_2[b]	AQL[b]	k_2[b]	AQL[b]	k_2[b]	AQL[b]
2	8.629	28.70%	11.109	21.56%	160.441	nearly 0%
5	2.597	16.68%	3.191	10.01%	8.291	0.013%
10	2.112	13.89%	2.564	7.79%	4.907	0.134%
15	1.981	12.97%	2.395	7.10%	4.170	0.239%
20	1.916	12.50%	2.312	6.74%	3.835	0.314%

[a] The unit "%" was not omitted for clear discernment of the AQLs from the tolerance limit factors.
[b] These two-sided tolerance limit factors and AQLs are the ones derived based on the assumption that the population mean coincides with the center of the specification range.

mean coincides with the center of the specification range. In this case, the two-sided tolerance limit factor is $k_2 = 2.597$ from Table 2. To obtain its corresponding AQL, we should first replace the confidence level $(1-\alpha) = 0.90$ with $(1-\alpha) = 0.95$. After then, we rewrite equation (7) to determine its two-tailed critical value K'_p as follows.

$$\alpha = 2 \int_0^{aK'_p\sigma/k_2} 2s'\phi\left(s'^2\right)\left\{\int_\mu^{(\mu+K'_p\sigma)-k_2s'/a} \Phi(\overline{x})d\overline{x}\right\}ds',$$

$$0.05 = 2 \int_0^{0.4\times 5K'_p/2.597} 2s'\phi\left(s'^2\right)\left\{\int_{500}^{(500+5K'_p)-2.597s'/0.4} \Phi(\overline{x})d\overline{x}\right\}ds'.$$

When rewriting, $a = (n-1)^{1/2}/\sigma = (5–1)^{1/2}/5 = 0.4$, $\mu = 500$, and $k_2 = 2.597$ were substituted into equation (7). Now, both by "division quadrature" and by "trial and error", we obtain $K'_p = 1.38253$. This two-tailed critical value is matched to the proportion $p = 0.8332$. Therefore, the AQL corresponding to $k_2 = 2.597$ is $(1-p)\times 100 = 16.68\%$. In this way, the AQLs corresponding to the one-sided and two-sided tolerance limit factors can be obtained. Table 7 shows some AQLs thus obtained. The unit "%" is omitted in practical uses of AQLs, however, it is not omitted for clear discernment of the AQLs from the tolerance limit factors in this article.

In Table 7, although two corresponding factors k_1 and k_2 are distinctly different from each other, the difference between the two AQLs matched to the two factors k_1 and k_2 is very small. For example, the AQL in the one-sided test of $n = 5$, $(1-\alpha) = p = 0.90$, and $k_1 = 2.742$ is 16.66% and it is nearly equal to the AQL 16.68% in the two-sided test of $n = 5$, $(1-\alpha) = p = 0.90$, and $k_2 = 2.597$. As another particular observation, in the case of $(1-\alpha) = p = 0.99$, as the sample size n increases, the AQL also increases. In contrast, in the case of $(1-\alpha) = p = 0.90$, the opposite phenomenon is seen.

The AQLs shown in Table 7 are based on the factors k_1 and k_2. However, the practical AQLs should be used in practical applications in the same manner as $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$ are used in practical applications. Table 8 presents some practical AQLs corresponding to $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$. It is common that people estimate the quality level of a lot from the AQL assigned to the sampling inspection. When establishing a sampling inspection plan, it is recommended that the fixed confidence level "$(1-\alpha) = 0.95$" be adopted, because, in this case, simply the value "$(1-p)$" itself becomes the AQL corresponding to the planned sampling inspection.

5 Results and discussion

The factors generated by the new double integration method were investigated through Monte Carlo simulations and it was identified that they are correct and exact. During calculation and simulation, the one-sided tolerance limit factors were compared with their respective corresponding two-sided tolerance limit factors generated by the new method. As a result, a new fact was found. When the sample size n is small, some k_2 factors are smaller than their respective corresponding k_1 factors. As the proportion p and the confidence level $(1-\alpha)$ come closer to 1, this phenomenon becomes more apparent. Particularly, in Table 4, through direct comparison of the two-sided tolerance limit factors determined by the new method with

Table 8. Practical AQLs corresponding to $k_{1,\text{practical}}$ and $k_{2,\text{practical}}$.

One–sided						
Sample size n	$1-\alpha=0.90,\ p=0.90$		$1-\alpha=0.90,\ p=0.95$		$1-\alpha=0.99,\ p=0.99$	
	$k_{1,practical}$	AQL	$k_{1,practical}$	AQL	$k_{1,practical}$	AQL
2	10.252	28.86%[a]	13.089	21.99%	185.543	nearly 0%
5	2.742	16.66%	3.400	10.05%	8.939	0.011%
10	2.066	13.81%	2.568	7.75%	5.074	0.131%
15	1.867	12.88%	2.329	7.04%	4.222	0.240%
20	1.765	12.40%	2.208	6.68%	3.832	0.318%
Two–sided						
Sample size n	$1-\alpha=0.90,\ p=0.90$		$1-\alpha=0.90,\ p=0.95$		$1-\alpha=0.99,\ p=0.99$	
	$k_{2,practical}$	AQL	$k_{2,practical}$	AQL	$k_{2,practical}$	AQL
2	10.252[b]	28.86%[b]	13.089[b]	21.99%[b]	185.543[b]	nearly 0%[b]
5	2.742[b]	16.66%[b]	3.400[b]	10.05%[b]	8.939[b]	0.011%[b]
10	2.112	13.89%	2.568[b]	7.75%[b]	5.074[b]	0.131%[b]
15	1.981	12.97%	2.395	7.10%	4.222[b]	0.240%[b]
20	1.916	12.50%	2.312	6.74%	3.835	0.314%

[a] The unit "%" was not omitted for clear discernment of the AQLs from the tolerance limit factors.
[b] In case that k_1 is larger than its corresponding factor k_2, the AQL corresponding to $k_{1,\text{practical}}$ was selected as the practical AQL in the two-sided test (see Table 7).

the ones generated by the existing methods, we can see that the existing two-sided tolerance limit factors being used in industry are inaccurate. As an example, for $n=10$ and $(1-\alpha)=p=0.90$, the two-sided tolerance limit factor k_2 determined by the new method is 2.112. However, according to the existing methods, k_2 is 2.546. Here, it was already demonstrated by the simulation results in Table 3 that the factor k_2 $(=2.112)$ is correct and exact. Therefore, we can say that the factor k_2 $(=2.546)$ being currently used is inaccurate.

If the population mean μ does not coincide with the center of the specification range, the practical two-sided tolerance limit factor $k_{2,\text{practical}}$ is affected by the deviation magnitude of μ from the center. Even in this case, if the population mean is known, the factor $k_{2,\text{practical}}$ can be calculated by the new method. In Section 3, it was illustrated that the factor $k_{2,\text{practical}}$ thus calculated is always placed somewhere between its two corresponding factors k_1 and k_2. As an example, for the inspection parameters of $n=5$, $(1-\alpha)=p=0.95$, $L=490$, and $U=510$, if the population mean is known to be 503.4206, in this case, we can calculate the practical factor $k_{2,\text{practical}}$ $(=4.156)$ by the new method, and hence, we can identify that the practical factor lies between its two corresponding factors k_1 $(=4.203)$ and k_2 $(=3.917)$. However, inspectors have to conduct the sampling inspection, not knowing the mean and variance of the normal population to be tested, and hence cannot calculate $k_{2,\text{practical}}$ directly. Therefore, $k_{2,\text{practical}}$ should be selected from the two corresponding factors k_1 and k_2 in a proper manner. If k_2 is smaller than k_1, k_1 (not k_2) should be used as substitute for $k_{2,\text{practical}}$. Unlike k_2, k_1 can always be used as substitute for $k_{1,\text{practical}}$.

The AQL is regarded as a common, familiar index representing the expected quality level of a product lot from the aspect of statistical quality control by sampling inspection. Therefore, it is necessary to match the tolerance limit factors to their respective corresponding AQLs. Fortunately, the AQL can also be derived by double integration of a joint probability density function of normal distribution and chi-square distribution in the same manner as used to generate the tolerance limit factor itself. In the AQL Tables produced by the new method, although two corresponding factors k_1 and k_2 are distinctly different from each other, the AQL matched to k_1 is nearly (but not exactly) equal to that matched to k_2. As an example, in Table 7, for $n=5$ and $(1-\alpha)=p=0.90$, the factor k_1 and its corresponding AQL determined by the new integration method are 2.742 and 16.66% respectively. In addition, the factor k_2 and its corresponding AQL determined in the same way are 2.597 and 16.68% respectively. Therefore, we can see that although the two factors k_1 and k_2 are distinctly different from each other, the two AQLs are nearly equal. As another interesting observation, in the case of $p=(1-\alpha)=0.99$, as the sample size n increases, the AQL also increases. In contrast, in the case of $p=(1-\alpha)=0.90$, the opposite phenomenon is seen. Especially, in case that $(1-\alpha)$ is 0.95, the value "(1–p)" itself can be regarded as the AQL corresponding to the tolerance limit factor. Therefore, it is recommended that when establishing a sampling inspection plan, the fixed confidence level "$(1-\alpha)=0.95$" be reflected in the plan.

6 Conclusion

The tolerance limit factors are important to conducting the statistical quality control by sampling inspection. The methodology for generating the one-sided tolerance limit factors was established in early 1960s. The factors generated by Owen [11] and Wheeler [12] are correct and exact, and are being fully utilized in industry. However, in the case of two-sided, any correct methodology has not been established yet, and hence the inaccurate two-sided tolerance limit factors are being currently used. In addition, it seems that conceptual understanding needed for applications of the two-sided tolerance limit factors is still insufficient.

To resolve such problems, this article introduced a new double integration method for generating correctly the one-sided and two-sided tolerance limit factors for normal populations along with a guidance for correct use of them. The new method generates the factors, not relying on the noncentral t-distribution, by double integration of a joint probability density function with respect to "s" (standard deviation) and "$\overline{x}$" (sample mean). Even AQLs corresponding to the factors can be calculated in the same way. In this article, the factors thus generated were investigated through Monte Carlo simulations and it was identified that they are correct and exact.

In the nuclear fuel fabrication plants, great numbers of fuel pellets are manufactured in the pellet production process every day. Each cylindrical pellet is approximately 5 grams in weight and approximately 1 cm in length. After the production of each lot, from 5 to 50 pellets as per quality inspection item are randomly sampled from the lot, and then pellet's diameter, length, density, etc. are precisely measured for quality control and inspection. However, in order to decide "conformance" or "non-conformance" for each of the pellet's quality inspection items (i.e., diameter, length, density, etc.), the inaccurate two-sided tolerance limit factors have been used till now. Now we know that those factors being currently used in the fabrication plants are the ones unduly overestimated by the existing methods.

In addition, it is not easy for the inspectors to use other special one-sided and two-sided tolerance limit factors than the ones that are frequently used in practical applications and can be obtained directly from the statistical handbooks. As an example, let's assume that the inspectors need the one-sided or two-sided tolerance limit factor to be used specifically for "$n=53$ and $p=(1-\alpha)=0.88$". In this special case, the inspectors cannot easily obtain the needed factor. The reason is that any statistical handbook cannot contain all the factors, therefore, the books contain only the factors to be frequently used. Moreover, in most cases, the inspectors cannot calculate directly the needed factor, because the existing methods are difficult and complicated to handle. Additionally, even in cases where the needed factors are found in the handbooks, the inspectors should be satisfied with the simple use of the factors only for inspection, not knowing the AQLs corresponding to those factors; because there have been no means of matching the factors to the AQLs till now.

However, from now on, if necessary, the inspectors or engineers in industry can directly generate the exact one-sided and two-sided tolerance limit factors, including the AQLs corresponding to the factors, by using the new double integration method introduced in this article. If anyone becomes accustomed to the use of this new method, it will presumably take a few minutes to generate a factor needed. Finally, we hope that this new method is widely disseminated and hence numerous inspectors performing the sampling inspection practically use the tolerance limit factors and AQLs generated by the new method.

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