Testing capability indices for one-sided processes with measurement errors

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1 Introduction

A common way to summarize the process performance is by using the process capability indices which provide information with respect to engineering specification. The two capability indices which have been widely used in manufacturing industry are $C_p$ and $C_{pk}$. The $C_p$ index measures only the potential of a process to produce acceptable product, and does not take into account where the process is centred. $C_{pk}$ index is computed using both location and dispersion information about the process. However, the indices $C_p$ and $C_{pk}$ consider only the process mean and standard deviation, but not the proximity of the process mean to the target value of the process characteristic. Chan et al. [1] proposed the $C_{pm}$ index to consider the proximity of the process mean to the target value. The $C_{pmk}$ index suggested by Pearn et al. [2] is a combination of $C_{pk}$ and $C_{pm}$, so it has the advantages of both $C_{pk}$ and $C_{pm}$. For a process with lower and upper tolerance limits $LSL$ and $USL$, and a target $T$ set to the midpoint $m = (LSL + USL)/2$ of the tolerance interval, Vännman [3] constructed a unified superstructure for the four previous basic indices which can be defined as

$$C_p(u,v) = \frac{d - u |\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where $d = (USL - LSL)/2$ is the half-length of the tolerance interval, $\mu$ is the mean, $\sigma$ is the standard deviation, and $u$ and $v$ are two non-negative parameters. Although cases with symmetric tolerances ($T = m$) are common in practical situations, cases with asymmetric tolerances ($T \neq m$) often occur in manufacturing industry. In general asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other. In this situation Chen and Pearn [4] suggested to use the family

$$C_p^u(u,v) = \frac{d^* - u A^*}{3\sqrt{\sigma^2 + v A^2}},$$

where $A = \max \{d(\mu - T)/D_u, d(T - \mu)/D_l\}$, $A^* = \max \{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, and $d^* = \min \{D_u, D_l\}$. Grau [5] suggests choosing parameters $u$ and $v$ according to the importance the user attaches to the process centring and/or to the percentage of non-conforming, which are the most important criteria to measure the process performance.

This paper considers the situation where a shift of the mean from the target in one direction appears less serious to the user, so that he is induced to define only one tolerance. Grau [6] suggests to the user to quantify the risk he considers the least serious when the mean shifts in the opposite direction from the target. When the risk is considered as $k$ times less serious, Grau [6] suggests using the families

$$C_p^u(u,v) = \frac{D_u - u A_u^*}{3\sqrt{\sigma^2 + v A_u^2}}, \text{ and } C_p^l(u,v) = \frac{D_l - u A_l^*}{3\sqrt{\sigma^2 + v A_l^2}},$$

for an upper and a lower tolerance, where $A_u^* = \max(\mu - T, (T - \mu)/k)$, and $A_l^* = \max((\mu - T)/k, T - \mu)$. Note that the letter $u$ used in superscript is an abbreviation of “upper”, and so too is independent of the first of the
two parameters \((u, v)\) of the indices family. The choice of
the constant \(k (\geq 1)\) is approximate, thus \(C_p^G(u, v)\) and
\(C_p'(u, v)\) have been constructed so that they are independent
of \(k\) when the mean shifts toward the tolerance limit. Note
that the choice \(k = 1\) comes to take into account similar
risks on the right and on the left. If for instance the risk
on the left is considered as nil, taking \(k\) finite is
sufficient. In this case when \(\mu < T\), \(C_p^G(u, v) = D_u/(3\sigma)\),
which is independent of \(\mu\), and which is nothing but the
potential capability \(C_p\). For \(C_p^G(u, v)\) and \(C_p'(u, v)\) Grau [7]
suggests choosing parameters \(u\) and \(v\) according to the
importance the user attaches to the process centring and/or
to the percentage of non-conforming.

The capability indices calculated from data obtained
from samples of manufactured items are not the true values
but only estimations. Therefore, a certain amount of
uncertainty, due to the sampling error, is necessarily
present in the evaluation of the process performance, and
in order to avoid erroneous interpretations, it is important
to know the statistical properties of the estimators. A fur-
ther source of uncertainty is given by the measurements
errors. Mittag [8] was the first author to link capability in-
dices and measurement errors. For the \(C_p, C_{pk}, C_{pu}, C_{pl},
C_{pm}, C_{pmk}\) and \(C_p^G(u, v)\) indices, Pearn and Liao [9–11],
Pearn et al. [12], Hsu et al. [13], Grau [14, 15] establish
reliable lower confidence bounds and reliable critical values
to estimate and test the process capability with gauge
measurement errors.

In this paper, we consider the one-sided process capa-
ibility indices \(C_p^G(u, v)\) and \(C_p'(u, v)\), and construct a criti-
cal value to test whether a process is capable when gauge
measurement errors are present.

This article is organized as follows. In Section 2, for
one-sided tolerance process we quickly report the main
results concerning the effects of gauge measurement errors
on theoretical capability indices \(C_p^G(u, v)\) and \(C_p'(u, v)\). In
Section 3 we give the sampling distribution of capability
indices in the presence of measurements errors. Sections 4
and 5 give critical values and adjusted critical values to
test process capability when gauge measurement errors
are noticed. Finally, a real example is presented in the
last section.

**2 The indices \(C_p^G(u, v)\), \(C_p'(u, v)\),
and the gauge measurement errors**

Suppose that the gauge measurement errors can be
described as a random variable \(M \sim N(0, \sigma_M^2)\). Mittag [8]
define the degree of error contamination as
\[
\tau = \frac{\sigma_M}{\sigma}.
\]
Suppose that \(X \sim N(\mu, \sigma^2)\) represents the relevant quality
characteristic of the manufacturing process. In practice
the observed variable \(G\) (with gauge measurement errors)
is measured rather than the true variable \(X\). It is further
assumed that \(X\) and \(M\) are additively linked according
to \(G = X + M\) and that \(X\) and \(M\) are stochastically
independent. Then we have \(G \sim N(\mu, \sigma^2 + \sigma_M^2)\)
and the empirical process capability indices \(C_p^G(u, v)\) and
\(C_p'(u, v)\) are obtained by formula (1) after substituting
\(\sigma_G\) for \(\sigma\).

\[
\begin{align*}
C_p^G(u, v) &= \frac{D_u - uA_u^*}{3 \sqrt{\sigma_G^2 + vA_l^2}}, \\
C_p'(u, v) &= \frac{D_l - uA_l^*}{3 \sqrt{\sigma_G^2 + vA_l^2}}
\end{align*}
\]

Obviously if \(\sigma_M = 0\), then the empirical process capability,
\(C_p^G(u, v)\) and \(C_p'(u, v)\), reduce to the basic indices,
\(C_p(u, v)\) and \(C_p'(u, v)\). Since the variation of the observed
data is greater than the variation of the original data,
the denominator of the indices \(C_p^G(u, v)\) and \(C_p'(u, v)\)
becomes larger and we would understated the true capa-
bility of the process if the empirical data are used. The
relation between \(C_p^G(u, v)\), \(C_p'(u, v)\), the true process
capability, and \(C_p^G(u, v)\), \(C_p'(u, v)\), the empirical process
capability, can be expressed as

\[
\begin{align*}
C_p^uG(u, v) &= \frac{1 + \varepsilon vu^2}{\sqrt{1 + \tau^2 + v\xi u^2}} \cdot C_p^u(u, v), \\
C_p^lG(u, v) &= \frac{1 + \varepsilon vl^2}{\sqrt{1 + \tau^2 + v\xi l^2}} \cdot C_p^l(u, v)
\end{align*}
\]

where \(\xi^u = \max (\xi, -\xi/k)\), \(\xi^l = \max (\xi/k, -\xi)\) and \(\xi =
(\mu - T)/\sigma\). It is clear that the ratio \(C_p^lG(u, v)/C_p^u(u, v)\) is
a decreasing function of \(\tau\). Then the measurements errors
understate the true theoretical capability of the process.
However, the theoretical capability is unknown and is
estimated from sample data. Thus, in the following sections
we deal with the effects of measurement errors on the be-
havior of estimated capability indices.

**3 Sampling distribution of \(\hat{C}_p^uG(u, v)\)
and \(\hat{C}_p^lG(u, v)\)**

Subsequently we will only develop the case of an upper
tolerance and without difficulty will deduce the similar
results in the case of a lower tolerance.

Let \(X_1, X_2, \ldots, X_n\) be a random sample from a
normal distribution with mean \(\mu\) and variance \(\sigma^2\)
measuring the characteristic of the process. To estimate the
\(C_p^u(u, v)\) and \(C_p'(u, v)\) indices, we replace \(\mu\) and \(\sigma^2\)
by their conventional estimators \(\bar{X} = (\sum_{i=1}^n X_i)/n\), and
\(S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n\). Thus we define the estimators
\(\hat{C}_p^uG(u, v)\) and \(\hat{C}_p^lG(u, v)\) as

\[
\begin{align*}
\hat{C}_p^uG(u, v) &= \frac{D_u - uA_u^*}{3 \sqrt{S^2 + vA_l^2}}, \\
\hat{C}_p^lG(u, v) &= \frac{D_l - uA_l^*}{3 \sqrt{S^2 + vA_l^2}}
\end{align*}
\]
where \( \hat{A}_* \) = \( \max (\bar{X} - T, (T - \bar{X}) / k) \) and \( \hat{A}_t \) = \( \max ((\bar{X} - T) / k, T - \bar{X}) \).

Under the assumption of normality, and for \( (u, v) \neq (0, 0) \), Grau [7] gives the cumulative distribution function of \( C_{pu}^u(u, v) \) which can be rewritten as

\[
F_{C_{pu}^u(u, v)}(x) = 1 - \int_0^{K(x)} H(t, x) dt, \quad x > 0,
\]

where \( K(x) = B_u \sqrt{\pi} / (u + 3x \sqrt{\pi}) \), \( B_u = D_u / \sigma = 3C_{pu}^u \) with \( C_{pu}^u = C_{pu}^u(0, 0) \),

\[
H(t, x) = F_K \left( \frac{(B_u \sqrt{\pi} - ut)^2 - u^2}{3u^2} \right) f_Z(t), \quad \text{with} \quad f_Z(t) = \phi(t - \sqrt{n} \xi) + k \phi (kt + \sqrt{n} \xi).
\]

For \( C_{pu}^u(u, v) \), we obtain similar results replacing \( B_u \) by \( B_1 = D_1 / \sigma = 3C_{pu}^u \), \( B_1^u = C_{pu}^u(0, 0) \), and \( f_Z(t) \) by \( f_{Z1}(t) = k \phi (kt - \sqrt{n} \xi) + \phi (t + \sqrt{n} \xi) \).

Thus for \( (u, v) \neq (0, 0) \), the cumulative distribution of \( C_{pu}^u(u, v) \) is

\[
F_{C_{pu}^u(u, v)}(x) = 1 - \int_0^{K_G(x)} H_G(t, x) dt, \quad x > 0,
\]

where \( K_G(x) = B_G \sqrt{\pi} / (u + 3x \sqrt{\pi}) \), \( B_G = D_u / \sigma_G = 3C_{pu}^G \) with \( C_{pu}^G = C_{pu}^G(0, 0) \),

\[
H_G(t, x) = F_K \left( \frac{(B_G \sqrt{\pi} - ut)^2 - u^2}{3u^2} \right) f_{Z_G}(t), \quad \text{with} \quad f_{Z_G}(t) = \phi(t - \sqrt{n} \xi_G) + k \phi (kt + \sqrt{n} \xi_G).
\]

For \( C_{pu}^G(u, v) \), we obtain similar results replacing \( B_G \) by \( B_1 = D_1 / \sigma_G = 3C_{pu}^G \) with \( C_{pu}^G = C_{pu}^G(0, 0) \), and \( f_{Z_G}(t) \) by \( f_{Z1}(t) = k \phi (kt - \sqrt{n} \xi_G) + \phi (t + \sqrt{n} \xi_G) \).

### 4 Capability testing based on \( \hat{C}_{pu}^G(u, v) \) or \( \hat{C}_{pu}^{G}(u, v) \)

To determine whether a given process meets the preset capability requirement, we consider the following statistical hypotheses testing:

- \( H_0 : C_{pu}^u(u, v) \leq c \) Process is not capable;
- \( H_1 : C_{pu}^u(u, v) > c \) Process is capable;

where \( c \) is the required process capability. If the calculated process capability \( \hat{C}_{pu}^u(u, v) \) is greater than the critical value \( c_0 \), we reject the null hypothesis and conclude that the process is capable with error \( \alpha \), which is the chance of incorrectly concluding an incapable process as capable. Given values of \( c \) and \( \alpha \), from (4) the critical value \( c_0 \) can be determined by solving the equation \( \alpha = P(\hat{C}_{pu}^u(u, v) > c_0 | C_{pu}^u(u, v) = c) = \int_0^{K(c_0)} H(c_0, t) dt \), where from (A.9) in Appendix A, \( B_u = 3C_{pu}^G = 3 \sqrt{1 + \beta} c + u \xi_c \). However, since the process parameters \( \mu \) and \( \sigma \) are unknown, parameters \( \xi \) and therefore \( \xi_u \) are unknown. We replace \( \xi \) and \( \xi_u \) by the observation of the natural estimators \( \xi = (X - T) / S \) and \( \xi_u = \max (\xi - \xi / k) \). Note that we use the same symbols \( \xi \) and \( \xi_u \) for the estimators as well as the estimations.

In the presence of measurements errors, \( C_{pu}^G(u, v) \) is measured rather than the true variable \( C_{pu}^u(u, v) \). Thus the \( \alpha \)-risk denoted by \( \alpha_G \) is defined as \( \alpha_G = P(C_{pu}^u(u, v) > c_0 | C_{pu}^G(u, v) = c) \). Earlier discussions indicate that we underestimate the real process capability using \( C_{pu}^G(u, v) \) instead of \( C_{pu}^u(u, v) \). The probability of \( C_{pu}^G(u, v) \) being greater than \( c_0 \) will be less important than by using \( C_{pu}^u(u, v) \). Thus \( \alpha_G \), the \( \alpha \)-risk using \( C_{pu}^G(u, v) \) to estimate \( C_{pu}^u(u, v) \) is less than \( \alpha \), using \( C_{pu}^u(u, v) \) when estimating \( C_{pu}^G(u, v) \). To illustrate the performance of \( \alpha_G \) versus \( \tau \), we have considered the particular case \( u = 0.5, v = 1.5, k = 3, c = 1 \) and \( 1.5, \alpha = 0.05 \), and have plotted curves \( \alpha_G \) for various \( C_{pu}^G \) values. The critical value \( c_0 \) is obtained from (4) where \( \xi \) is defined from (A.2) and \( \xi_u \) is defined from (A.14) as

see equation above.

Then from (5) we compute \( \alpha_G = \int_0^{K_G(c_0)} H_G(c_0, t) dt \), where \( C_{pu}^G \) and \( \xi_G \) are obtained from (A.12) and (A.5).

Figure 1 plots \( \alpha_G \) versus \( \tau \) \( \in [0, 1] \) with \( n = 30, 50, 70, 100, 150 \) for \( C_{pu}^G = c(0.5)(c + 1) \). Note that for \( \tau = 0, \alpha_G = \alpha \). In Figure 1, \( \alpha_G \) decreases as \( \tau \) or \( n \) increases, and the decreasing rate is more significant with large \( c \) and small \( C_{pu}^G \).

We now consider the power of the test, the chance of correctly concluding a capable process as capable. It can
Fig. 1. Plots of $\alpha_G$ versus $\tau$ with $n = 30, 50, 70, 100, 150$ (top to bottom), $u = 0.5$, $v = 1.5$, $k = 3$, $\alpha = 0.05$ for (a) $c = 1$, $C_p = 1$; (b) $c = 1$, $C_p = 1.5$; (c) $c = 1$, $C_p = 2$; (d) $c = 1.5$, $C_p = 1.5$; (e) $c = 1.5$, $C_p = 2$; (f) $c = 1.5$, $C_p = 2.5$. 

(a) $\xi = 0$ 
(b) $\xi > 0$ 
(c) $\xi < 0$ 
(d) $\xi = 0$ 
(e) $\xi > 0$ 
(f) $\xi < 0$
be computed as \( \pi(C_p(u, v)) = P(\hat{C}_p(u, v) \geq c_0 | C_p(u, v)) = \int_0^{\hat{K}_G(c_0)} H_G(c_0, t)dt \), with from (A.9) \( B_u = 3C_p = 3\sqrt{1 + c^\alpha c + u^\alpha u} \).

In the presence of measurements errors, the power of the test denoted by \( \pi_G \) is as follows \( \pi_G(C_p(u, v)) = P(C_p^u(u, v) > c_0 | C_p(u, v)) = \int_0^{\hat{K}_G(c_0)} H_G(c_0, t)dt \). To illustrate the performance of \( \pi_G \) versus \( \tau \), we consider the particular case \( u = 0.5, v = 1.5, k = 3 \) and plot the curves \( \pi_G \) for various \( C_p^u \) and \( C_p^u(u, v) \) values. In the previous equation, \( C_p^u, \xi^u, \xi \) and \( \xi_G \) are obtained from (A.12), (A.14), (A.2) and (A.5). Figure 2 plots \( \pi_G \) versus \( \tau \in [0, 1] \) with \( n = 50, \alpha = 0.05, c = 1.00 \) and 1.50, \( C_p^u(u, v) = c(0.20)(c+ 1) \), and \( C_p^u = C_p^u(u, v) + 0.33 \) and \( C_p^u(u, v) + 0.5 \). In Figure 2, we see that \( \pi_G \) decreases as \( \tau \) increases and the decreasing rate is more significant with large \( c \). The presence of measurement errors can have a very substantial effect on \( \pi_G \). For instance, for \( c = 1.5 \) and \( C_p^u = C_p^u(u, v) = 2.3 \) (Fig. 2d), \( \pi_G \) is approximately equal to 0 without measurement errors. However, when \( \tau = 1 \), \( \pi_G \) is approximately equal to 0.15 for \( \xi > 0 \) and approximately equal to 0.41 for \( \xi < 0 \).

5 Adjusted critical values

As we have seen in the previous section, the \( \alpha \)-risk and the test power decrease with measurement errors. The capability testing results would be misleading if the producers do not take account of the gauge measurements errors. Thus, in order to improve the test power, we revise the critical value, denoted by \( c^A_0 \), to satisfy \( c^A_0 < c_0 \). Let \( \alpha_A = P(C_p^G(u, v) > c^A_0 | C_p^G(u, v) = c) \) and \( \pi_A(C_p^G(u, v)) = P(C_p^G(u, v) > c^A_0 | C_p^G(u, v)) \), be the \( \alpha \)-risk and the test power using the adjusted critical value. Since \( c^A_0 < c_0 \), \( P(C_p^G(u, v) > c^A_0) \) is greater than \( P(C_p^G(u, v) > c_0) \), and both \( \pi_A \) and \( \alpha_A \) increase. To ensure that the \( \alpha \)-risk is within the pre-set magnitude, we set \( \alpha_A = \alpha \), and solve equation \( \alpha = P(C_p^u(u, v) > c^A_0 | C_p^u(u, v) = c) \) to obtain \( c^A_0 \). Thus from (5), we must solve

\[
\alpha = \int_0^{\hat{K}_G(c^A_0)} H_G(c_0^A, t)dt. \tag{6}
\]

Since the process parameters \( \mu \) and \( \sigma_G \) are unknown, then \( \xi_G \) is also unknown. Thus the previous equation involves the unknown parameters \( \xi_G \) and \( C_p^u \) or \( C_p^u \) from (A.12).

We replace \( \xi_G \) by the observed value \( \xi_{G1} \), and from (A.13) \( C_p^u \) can be obtained by

\[
C_p^u = \sqrt{1 + \tau^2} \left( \frac{1 + u^\xi_G^2}{\sqrt{1 + \tau^2 + u^\xi_G^2}} c + u^\xi_G^u / 3 \right), \tag{7}
\]

where \( \xi_G^u \) and \( \xi_G^u \) are defined from (A.3) and (A.7) as follows

\[
\xi_G^u = \max \left( \xi_G, -\xi_G/k \right),
\]

and \( \xi_G^u = \max \left( \xi_G^u, -\xi_G^u/k \right). \]

With that adjusted critical value \( c^A_0 \), we can calculate the test power noted down as \( \pi_A \). Let \( C_p^{u1} \) be the value of \( C_p^u \) the solution of equation (7). The test power can be calculated as follows

\[
\pi_A(C_p(u, v)) = P(C_p^{u1}(u, v) > c_0 | C_p(u, v), C_p = C_p^{u1})
= \int_0^{K_1(c^A_0)} H_1(c_0^A, t)dt,
\]

with

\[
K_1(x) = B_{G1}^G \sqrt{n}/(u + 3x\sqrt{n}),
B_{u1}^G = 3C_p^{u1G}/3c_1(u) \tag{A.12}.
\]

\[
H_1(x, t) = F_K \left( \frac{B_{G1}^G \sqrt{n} - ut}{3x} \right)^2 - ut^2 f_{Z^u}(t),
\]

\[
f_{Z^u}(t) = \phi - \sqrt{n}\xi_{G1} + k\phi \left( t - \sqrt{\xi_{G1}} \right) \xi_{G1},
\xi_{G1} = \begin{cases} kG_1^G & \text{if } \xi_{G1} > 0, \\ -kG_{-1}^G & \text{if } \xi_{G1} < 0. \end{cases}
\]

and \( \xi_{G1}^u \) is obtained substituting \( C_p^{u1} \) to \( C_p^u \) in (A.14). \( \xi_{G1} \) and \( \xi_{G1}^u \) have the same sign, but \( \xi_G^u \) is unknown. Thus we replace \( \xi_G \) by the observed value \( \xi_{G1} \), and replace \( \xi_{G1} \) by \( \xi_{G1}^u \) defined by \( \xi_{G1}^u = \begin{cases} \xi_{G1} & \text{if } \xi_{G1} > 0, \\ -\xi_{G1} & \text{if } \xi_{G1} < 0. \end{cases} \)

For a process with a lower tolerance, we obtain similar results replacing, \( C_p^{u1} \) by \( C_p^u \) the solution of equation

\[
C_p^l = \sqrt{1 + \tau^2} \left( \frac{1 + u^\xi_G^l}{\sqrt{1 + \tau^2 + u^\xi_G^l}} c + u^\xi_G^u / 3 \right), \tag{8}
\]

\[
\xi_G^l \text{ by } \xi_G^l, \text{ by } \xi_l \text{ by } \xi_l = \max \left( \xi_G^l/k, -\xi_G^l/1 + \tau^2 \right),
B_{u1}^G \text{ by } B_{u1}^G = 3C_p^{u1}/1 + \tau^2, \quad f_{Z^u}(t) = \phi \left( t - \sqrt{n}\xi_{G1} \right) + k\phi \left( t - \sqrt{n}\xi_{G1} \right) + \phi \left( t + \sqrt{n}\xi_{G1} \right),
\xi_{G1} = \begin{cases} kG_1^G & \text{if } \xi_{G1} > 0, \\ -kG_{-1}^G & \text{if } \xi_{G1} < 0. \end{cases}
\]

\[
\xi_{G1}^l = \xi_l / \sqrt{1 + \tau^2}, \text{ and } \xi_l^l \text{ is obtained substituting } C_p^{u1} \text{ to } C_p^u \text{ in (A.14).}
\]

A Maple program is developed in Appendix B to compute the adjusted critical value \( c^A_0 \) and plot the power test. Figure 3 plots \( \pi_A \) versus \( \tau \in [0, 1] \) with \( u = 0.5, v = 1.5, k = 3, n = 50, \alpha = 0.05, c = 1.00,1.50, \)
Fig. 2. Plots of $\pi_G$ versus $\tau$ with $u = 0.5$, $v = 1.5$, $k = 3$, $n = 50$, $\alpha = 0.05$, $C_p^u(u, v) = c(0.20)(c + 1)$ (bottom to top) for (a) $c = 1$ and $C_p^u = C_p^u(u, v)$; (b) $c = 1$ and $C_p^u = C_p^u(u, v) + 0.33$; (c) $c = 1$ and $C_p^u = C_p^u(u, v) + 0.5$; (d) $c = 1.5$ and $C_p^u = C_p^u(u, v)$; (e) $c = 1.5$ and $C_p^u = C_p^u(u, v) + 0.33$; (f) $c = 1.5$ and $C_p^u = C_p^u(u, v) + 0.5$. 
Fig. 3. Plots of $\pi_A$ versus $\tau$ with $u = 0.5$, $v = 1.5$, $k = 3$, $n = 50$, $\alpha = 0.05$, $C_p(u, v) = c(0.20)(c + 1)$ (bottom to top) for (a) $c = 1$ and $C_p^u = C_p^u(u, v)$; (b) $c = 1$ and $C_p^u = C_p^u(u, v) + 0.33$; (c) $c = 1$ and $C_p^u = C_p^u(u, v) + 0.5$; (d) $c = 1.5$ and $C_p^u = C_p^u(u, v)$; (e) $c = 1.5$ and $C_p^u = C_p^u(u, v) + 0.33$; (f) $c = 1.5$ and $C_p^u = C_p^u(u, v) + 0.5$. 
From the Maple program in Appendix B, we obtain that depends on the selected sample and does not make more than 1.33. However this value is a specific point estimate between the target and the lower tolerance limit (Grau [7]).

The objective of the Arkema company is to use a capability index with a clear interpretation in terms of centring and non-conforming. The selected index is \( C_p(C_{p}', C_{p}'') \), which is capable when the capability indices are higher than 1.33.

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In this work we have been concerned with the problem of gauge measurement errors when dealing with process capability indices for one-sided tolerances. Grau [6] introduces the \( C_p(C_{p}', C_{p}'') \) indices to cover the processes with one-sided tolerances. The quality of a process is defined by the centring process and the proportion of non-conforming manufactured items. Grau [7] gives the elements that will allow the user to choose the pair \((u, v)\) that will meet his needs as best as possible. In this paper we show that gauge measurement errors have a significant impact on the estimation of a process capability. The estimator evaluating the capability by using the sample data contaminated with gauge measurement errors severely underestimates the \(\alpha\)-risk, resulting in a smaller testing power. We suggest an adjusted critical value to improve the accuracy of the capability assessment. For practical purpose we give a Maple program helpful to the users for their factory applications. Finally a real example in a company producing polymer granulates is used to assess the interest of the approach.

## Appendix A:

- We have

\[
\xi^u = \max(\xi, -\xi/k), \quad \xi^l = \max(\xi/k, -\xi) \quad (A.1)
\]

\[
\xi = \begin{cases} 
\xi^u & \text{if } \xi > 0 \\
-\xi^l & \text{if } \xi < 0
\end{cases} \quad \xi = \begin{cases} 
\xi^u & \text{if } \xi > 0 \\
-\xi^l & \text{if } \xi < 0
\end{cases} \quad (A.2)
\]

\[
\xi_G = \max(\xi_G, -\xi_G/k), \quad \xi_G = \max(\xi_G/k, -\xi_G) \quad (A.3)
\]

\[
\xi_G = \begin{cases} 
\xi^u_G & \text{if } \xi_G > 0 \\
-\xi^l_G & \text{if } \xi_G < 0
\end{cases} \quad \xi_G = \begin{cases} 
\xi^u_G & \text{if } \xi_G > 0 \\
-\xi^l_G & \text{if } \xi_G < 0
\end{cases} \quad (A.4)
\]

Since \( \xi = (\mu - T)/\sigma \) and \( \xi_G = (\mu - T)/\sigma_G \), we have

\[
\xi_G = \xi/\sqrt{1 + \tau^2} \quad (A.5)
\]

\[
\xi_G = \xi_G + \sqrt{1 + \tau^2} \quad (A.6)
\]

- From (A.1) and (A.6), we have

\[
\xi^u = \max\left(\xi_G + \sqrt{1 + \tau^2} - \xi_G \sqrt{1 + \tau^2}/k, 0\right) \quad (A.7)
\]

- From (A.3), (A.5) and (A.1), we have

\[
\xi_G^u = \xi^u/\sqrt{1 + \tau^2}, \quad \xi_G^l = \xi^l/\sqrt{1 + \tau^2} \quad (A.8)
\]

- Since \( \xi^u = X^u_0/\sigma \) and \( \xi^l = X^l_0/\sigma \), from (1) we have

\[
C_p(u, v) = c(0.20)(c + 1) \quad (A.9)
\]

\[
C_p = C_p(u, v) \quad (A.10)
\]

\[
C_p' = C_p(u, v) + 0.33 \quad (A.11)
\]

\[
C_p'' = C_p(u, v) + 0.5 \quad (A.12)
\]
\[ \xi^u = \frac{-C_p^u u/3 + \sqrt{(C_p^u)^2 (u/3)^2 + \left((C_p^u)^2 - (C_p^u (u, v))^2\right) \left(v (C_p^u (u, v))^2 - (u/3)^2\right)}}{v (C_p^u (u, v))^2 - (u/3)^2} \]
\[ \xi^l = \frac{-C_p^l u/3 + \sqrt{(C_p^l)^2 (u/3)^2 + \left((C_p^l)^2 - (C_p^l (u, v))^2\right) \left(v (C_p^l (u, v))^2 - (u/3)^2\right)}}{v (C_p^l (u, v))^2 - (u/3)^2} \] (A.14)

\[ C_p^u = \sqrt{1 + v \xi^u} C_p^u (u, v) + u \xi^u / 3, \]
\[ C_p^l = \sqrt{1 + v \xi^l} C_p^l (u, v) + u \xi^l / 3 \] (A.9)

- In the same manner, from (2) we have
\[ C_p^{\text{BG}} = \sqrt{1 + v \xi^{\text{BG}}} C_p^{\text{BG}} (u, v) + u \xi^{\text{BG}} / 3, \]
\[ C_p^{\text{LG}} = \sqrt{1 + v \xi^{\text{LG}}} C_p^{\text{LG}} (u, v) + u \xi^{\text{LG}} / 3 \] (A.10)

- From (A.10) and (3) we have
\[ C_p^{\text{BG}} = C_p^{\text{BG}} (0, 0) \text{ and } C_p^{\text{LG}} = C_p^{\text{LG}} (0, 0), \text{ from (3)} \]
\[ C_p^{\text{BG}} = C_p^{\text{BG}} \sqrt{1 + \tau^2}, \]
\[ C_p^{\text{LG}} = C_p^{\text{LG}} \sqrt{1 + \tau^2} \] (A.12)

- From (A.12) and (A.11), we have
\[ C_p^u = \sqrt{1 + \tau^2} \left( \frac{1 + v \xi^u \sqrt{1 + v \xi^u}}{\sqrt{1 + \tau^2 + v \xi^u}} C_p^u (u, v) + u \xi^u / 3 \right), \]
\[ C_p^l = \sqrt{1 + \tau^2} \left( \frac{1 + v \xi^l \sqrt{1 + v \xi^l}}{\sqrt{1 + \tau^2 + v \xi^l}} C_p^l (u, v) + u \xi^l / 3 \right) \] (A.13)

- From (1), by a proof identical to Grau [15], we obtain
see equation (A.14) above

**Appendix B:**

The Maple Program to compute the critical value \( c_0^A \)

When the function \( H_G \) takes values that are too low, the Maple software is sometimes unable to perform the calculation of the integral of \( H_G \). This is the reason why we limit the domain of integration \([0; K_G]\) to the domain \([K_0; K_1]\) on which \( H_G \geq 10^{-5} \). This does not change the value of the integral for the required precision.

**Algorithm**

Step 1. Read \( u, v, k, \tau, n, \alpha, c, \xi_G, B \) (the tolerance limit)

Step 2. Compute \( \xi^u \) or \( \xi^l \) from (A.7), \( \xi^u_G \) or \( \xi^l_G \) from (A.3), \( C_p^u \) or \( C_p^l \) from (7) or (8), \( C_p^{\text{BG}} \) or \( C_p^{\text{LG}} \) from (A.12).

Step 3. Compute \( H_G, K_G \).

Step 4. Find the adjusted critical value \( c_0^A \) from (6).

Step 4.1. Find \( K_0 \) a lower integration bound.

Step 4.2. Find \( K_1 \) an upper integration bound.

Step 5. Print the adjusted critical value.

Step 6. Plot the power test.

**Maple program**

restart:with(stats):
#1. Read u, v, k (risk), tau (error contamination), n (number of observations), alpha (the alpha-risk), c (threshold value for a capable process), xG (estimation of (muG-T)/sigmaG), B (the bound, L for Lower, U for Upper) u:=0.6:v:=0.2:k:=3:tau:=0:n:=112:alpha:=0.05:c:=1.33:
xG:=0.854:B:=L:
#2. Compute xB, xBG, CpB, CpBG if (B=L) then xBG:=max(xG/k,-xG) else xB:=max(xG*sqrt(1+tau^2)/k,-xG*sqrt(1+tau^2)/k) fi:
if (B=L) then xB:=max(xG/k,-xG) else xBG:=max(xG,-xG) fi:
CpB:=sqrt(1+tau^2)*dist[1+t+2]*sqrt(1+v*xBG^2)*dist[1+t+2]/sqrt(1+t+2+v*xB^2)*c+u*xBG/3):
CpBG:=CpB/dist[1+t+2]:
#3. Compute H and K
F:=(c0,t)->dist[1+t+2]*sqrt(1+v*xBG^2)*dist[1+t+2]/sqrt(1+t+2+v*xB^2)*c+u*xBG/3):
CpBG:=CpB/dist[1+t+2]:
#4. Compute c0 the critical value
k1:=0:k2:=5:9:
for i from 1 to 100
do k2:=(k1+k2)/2:evalf(k2):
# 4.1.Find K0, lower integration bound
K0:=0:c1:=evalf(H(c0,K0)):
if z1<1e-8 then for j from 1 to trunc(Kl)
do z1:=evalf(H(c0,j));
if z1>1e-8 then for h from j-0.9 by 0.1 to j
do z1:=evalf(H(c0,h));
if z1>1e-8 then K0:=h-0.1:break fi:
od:
break fi:
od:
fi:
# 4.2.Find K1, upper integration bound
K1:=evalf(Kl)-0.00001: z2:=evalf(H(c0,K1)):
if z2<1e-8 then for j from K1 by -1 to 0
do z2:=evalf(H(c0,j));
if z2>1e-8 then for h from j+0.9 by -0.1 to j
do z2:=evalf(H(c0,h));
if z2>1e-8 then K1:=h+0.1:break fi:
od:
break fi:
od:
fi:
if z1<1e-8 and z2<1e-8 then y:=0:
else y:=evalf(int(H(c0,t),t=evalf(K0),evalf((K0+K1)/2)))+evalf(int(H(c0,t),t=evalf((K0+K1)/2),evalf(K1))):
fi:
if y<alpha then k3:=k2 else k1:=k2 fi:
if abs(y-alpha)<0.00001 then break fi:
od:
# 5. Result
printf(“the critical value is %.3f with %.2f per cent risk”,c0,100*alpha):
# 6. Plot the power test
xB1:=(-CpB*u/3+sqrt(CpBˆ2*(u/3)ˆ2+(CpBˆ2-CpBuvˆ2)*(v*CpBuvˆ2-(u/3)ˆ2)))/(v*CpBuvˆ2-(u/3)ˆ2):
xBG1:=xB1/sqrt(1+tauˆ2):
if (B=L) then xG1:=piecewise(xG>0,k*xBG1,-xBG1)
else xG1:=piecewise(xG>0,xBG1,k*xBG1) fi:
if (B=L) then fB:=t-k*stats[statevalf,pdf,normald](k*t-xG1*nˆ.5)+stats[statevalf,pdf,normald](t+xG1*nˆ.5)
else fB:=t-stats[statevalf,pdf,normald](t-xG1*nˆ.5)+k*stats[statevalf,pdf,normald](k*t+xG1*nˆ.5) fi:
pw:=c0->int(F(c0,t)*fB(t),t=0..K1):
plot(pw(c0),CpBuv=c..CpB,title=“test power”):

References