Testing capability indices for manufacturing processes with asymmetric tolerance limits and measurement errors

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Abstract. Most research works related to process capability indices assume no gauge measurement errors. However, such an assumption inadequately reflects real situations even when advanced measuring instruments are employed. If we do not take into account these errors, conclusions drawn from process capability are therefore unreliable. In this paper we study the sampling distribution of capability indices $C_p''(u,v)$ in the presence of measurements errors, and when small subsamples data are collected from past "in-control". We show that using a critical value without taking into account these errors, severely underestimates the α-risk which causes a less accurate testing capacity. To improve the results we suggest the use of an adjusted critical value, and we give a Maple program to get it. An example in a nougat manufactory is presented to illustrate this approach.

Keywords: Process capability indices; gauge measurement error; asymmetric tolerances; critical value; power test

1 Introduction

Process capability indices, which provide numerical measures on whether a process meets the capability requirement preset in the factory, have been widely applied to the industry for evaluating manufacturing performance. Recent research in numerous articles and several books such as Kotz and Johnson [1], Bothe [2], Kotz and Lovelace [3], or Pearn and Kotz [4] have focussed on the study of these capability indices. The original process capability index is C_p, which is designed to provide a measure of potential ability to meet requirements. The major weakness of this index lies in the fact that it does not consider the mean of the process. Kane [5] considered the C_{pk} index in order to show the influence of a shift of the process mean on the ability of the process to produce products within the tolerance values. The C_{pm} index suggested by Chan et al. [6] involves the variation of production items with respect to the target value and the tolerance limits that are preset in the factory. Combining the advantages of these previous indices, Pearn et al. [7] introduced the C_{pmk} index. For a process with lower and upper tolerance limits LSL and USL, and a target T set to the midpoint $m = (LSL + USL)/2$ of the tolerance interval, Vännman [8] constructed a unified superstructure for the

four previous basic indices which can be defined as

$$C_p(u,v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where $d = (USL - LSL)/2$ is the half-length of the tolerance interval, μ is the mean, σ is the standard deviation, and u and v are two non-negative parameters. It is easy to verify that $C_p = C_p(0,0)$, $C_{pk} = C_p(1,0)$, $C_{pm} = C_p(0,1)$, and $C_{pmk} = C_p(1,1)$.

A process is said to have symmetric tolerances if the target value T is set to the midpoint m of the tolerance interval, that is to say $T = m$. Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances often occur in manufacturing industry. In general asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other. Nevertheless, asymmetric tolerances can also arise in those situations where the tolerances are symmetric to begin with, but the process distribution following a non-normal distribution, the data are transformed to achieve approximate normality. If indices $C_p(u,v)$ are well adapted to the case of symmetrical tolerances, they have some undesirable properties when the tolerances are asymmetrical, $T \neq m$ (see Boyles [9]). To overcome the problems with asymmetric tolerances, and to generalize the family $C_p(u,v)$ to the case $T \neq m$, Chen and Pearn [10] suggested using the family

$$C_p''(u,v) = \frac{d^* - uA^*}{3\sqrt{\sigma^2 + vA^2}}, \tag{1}$$

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where $A = \max\{d(\mu - T)/D_u, d(T - \mu)/D_l\}$, $A^* = \max\{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, and $d^* = \min\{D_u, D_l\}$. Grau [11] suggests choosing parameters u and v according to the importance the user attaches to the process yield, or to the process centring, which are the most important criteria to measure the process performance.

In practice, process capability indices are unknown and are estimated from data obtained from samples of manufactured items. Therefore, a certain amount of uncertainty, due to the sampling error, is necessarily present in the evaluation of the process performance. A further source of uncertainty is given by the measurements errors even with highly advanced measuring instruments. Montgomery and Runger [12,13] pointed out that quality of the collected data on the process characteristics mostly relies on the gauge accuracy. Levinson [14] and Mittag [15] were the first to quantify the percentage error on process capability indices evaluation in the presence of measurement errors. Bordignon and Scagliarini [16–18] study the statistical properties of the estimators of C_p, C_{pk}, and C_{pm} in the case of measurement errors. For the C_p, C_{pk}, C_{PU}, C_{PL}, C_{pm} and C_{pmk} indices, Pearn and Liao [19–21], Pearn et al. [22], Hsu et al. [23] present some statistical properties and establish reliable lower confidence bounds and reliable critical values to estimate and test the process capability with gauge measurements errors.

The purpose of this paper is to construct a critical value to test whether a process is capable when gauge measurement errors are present. This article is organized as follows. In Section 2, for processes with symmetrical or asymmetrical tolerance limits, we quickly report the main results concerning the effects of gauge measurement errors on theoretical capability indices. In Section 3 we give the sampling distribution of capability indices in the presence of measurements errors, and when small subsamples data are collected from past "in-control". Sections 4 and 5 give critical values and adjusted critical values to test process capability when gauge measurement error is present. Finally, a real example is presented in the last section.

2 The index $C_p''(u, v)$ and the gauge measurement errors

Suppose that the gauge measurement errors can be described as a random variable $M \square N(0, \sigma_M^2)$. Montgomery and Runger [12] expressed the gauge capability as

$$\lambda = \frac{6\sigma_M}{USL - LSL}.$$

In this paper, the gauge capability provided by the gauge manufacturing factory is assumed to be known. Suppose that $X \square N(\mu, \sigma^2)$ represents the relevant quality characteristic of the manufacturing process. In practice the observed variable G (with gauge measurement errors) is measured rather than the true variable X. It is further assumed that X and M are additively linked according to

$G = X + M$ and that X and M are stochastically independent. Then we have $G \square N(\mu, \sigma_G^2 = \sigma^2 + \sigma_M^2)$ and the empirical process capability index $C_p''^G(u, v)$ is obtained by formula (1) after substituting σ_G for σ.

$$C_p''^G(u, v) = \frac{d^* - uA^*}{3\sqrt{\sigma_G^2 + vA^2}}. \tag{2}$$

Obviously if $\sigma_M = 0$, then the empirical process capability $C_p''^G(u, v)$ reduces to the basic index $C_p''(u, v)$. Since the variation of the observed data is larger than the variation of the original data, the denominator of the index $C_p''^G(u, v)$ becomes larger and we would understate the true capability of the process if the empirical data G are used.

Let $\delta = (T - m)/d$ be the quantity that allows locating the position of T in the tolerance interval. We have $d/D_u = (1 - \delta)^{-1}$, $d/D_l = (1 + \delta)^{-1}$, and $d^*/d = 1 - |\delta|$. Assume that $C_p = C_p(0, 0)$ and $C_p'' = C_p''(0, 0)$. Since $\lambda C_p''/(1 - |\delta|) = \lambda C_p = \sigma_M/\sigma$, the relationship between the true process capability, $C_p''(u, v)$, and the empirical process capability, $C_p''^G(u, v)$, can be expressed as

$$C_p''^G(u, v) = \frac{\sqrt{1 + v\xi^{*2}}}{\sqrt{1 + \lambda^2 C_p''^2/(1 - |\delta|)^2 + v\xi^{*2}}} C_p''(u, v), \tag{3}$$

where $\xi^* = \max((1 - \delta)^{-1}\xi, -(1 + \delta)^{-1}\xi)$ and $\xi = (\mu - T)/\sigma$. It is clear that the ratio $C_p''^G(u, v)/C_p''(u, v)$ is decreasing function of λ and C_p''. Therefore the measurements errors underestimate the true theoretical capability of the process, especially when the potential capability C_p'' is large. However, the theoretical capability is unknown and is estimated from sample data. Thus, in the following sections we deal with the effects of measurement errors on the performance of estimated capability indices.

3 Sampling distribution of $\hat{C}_p''^G(u, v)$

A common practice of the process capability estimation in the manufacturing industry is to first implement a routine-based data collection program for monitoring the process stability. In order to analyze the past "in control" data, r subgroups with variable sample sizes n_i, $(X_{i1}, X_{i2}, \ldots, X_{in_i})$, are selected randomly from a stable process following a normal distribution $N(\mu, \sigma^2)$. Let $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$ and $S_i = \left[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \right]^{1/2}$ be the ith sample mean and the sample standard deviation, respectively, and $N = \sum_{i=1}^{r} n_i$ the total number of observations. We consider the following natural estimator of $C_p''(u, v)$ defined as

$$\hat{C}_p''(u, v) = \frac{d^* - u\hat{A}^*}{3\sqrt{S^2 + v\hat{A}^2}},$$

where $\hat{A} = \max\{(1 - \delta)^{-1}(\bar{\bar{X}} - T), (1 + \delta)^{-1}(T - \bar{\bar{X}})\}$, $\hat{A}^* = (d^*/d)\hat{A} = (1 - |\delta|)\hat{A}$, $\bar{\bar{X}} = \sum_{i=1}^{r} n_i \bar{X}_i/N$, and

$$\xi^* = \frac{-C_p''u(1-|\delta|)/3 + \sqrt{\left(C_p''\right)^2\left(u(1-|\delta|)/3\right)^2 + \left(\left(C_p''\right)^2 - c^2\right)\left(vc^2 - \left(u(1-|\delta|)/3\right)^2\right)}}{vc^2 - \left(u(1-|\delta|)/3\right)^2}.$$

$S^2 = \sum_{i=1}^r n_i S_i^2/N$. We define $K = NS^2/\sigma^2$, $Z = N^{1/2}(\bar{\bar{X}} - T)/\sigma$, and $Y = [\max\{(1-\delta)^{-1}Z, -(1+\delta)^{-1}Z\}]^2$. Then the estimator $\hat{C}_p''(u,v)$ can be rewritten as

$$\hat{C}_p''(u,v) = \frac{3\sqrt{N}C_p'' - (1-|\delta|)u\sqrt{Y}}{3\sqrt{K + vY}}.$$

Under the assumption of normality, K is distributed as χ_{N-r}^2, an ordinary central Chi-square distribution with $N - r$ degree of freedom, Z is distributed as $N(\sqrt{N}\xi, 1)$ and Y is distributed as $\chi_1^2(N\xi^2, (1-\delta)^{-1}, (1+\delta)^{-1})$, an unbalanced non-central chi-square distribution with one degree of freedom and noncentrality parameter $N\xi^2$ (Grau [24]). Applying the same technique used in Pearn et al. [25] to obtain the cumulative distribution of $\hat{C}_{pmk}'' = \hat{C}_p''(1,1)$, for $(u,v) \neq (0,0)$ we have

$$F_{\hat{C}_p''(u,v)}(x) = 1 - \int_0^{K(x)} H(x,t)dt, \quad \text{for} \quad x > 0 \quad (4)$$

where $K(x) = b\sqrt{N}/((1-|\delta|)u + 3x\sqrt{v})$, $b = 3C_p''$,

$H(x,t) = F_K((\frac{b\sqrt{N}-(1-|\delta|)ut}{3x})^2 - vt^2)f_Y(t)$, with $F_K(x)$ the cumulative distribution of K, $f_Y(t) = (1-\delta)\phi\left((1-\delta)t - \sqrt{N}\xi\right) + (1+\delta)\phi((1+\delta)t + \sqrt{N}\xi)$, and $\phi(x)$ the probability density of the standard normal distribution $N(0,1)$.

When there are gauge measurement errors, variable G is measured rather than the true variable X. r subgroups $(G_{i1}, G_{i2}, \ldots, G_{in_i})$ are selected randomly from a stable process following a normal distribution $N(\mu, \sigma_G^2)$. We denote $\bar{G}_i = \sum_{j=1}^{n_i} G_{ij}/n_i$, $S_{G_i} = [n_i^{-1}\sum_{j=1}^{n_i}(G_{ij} - \bar{G}_i)^2]^{1/2}$, $\xi_G = (\mu - T)/\sigma_G$, $\xi_G^* = \max((1-\delta)^{-1}\xi_G, -(1+\delta)^{-1}\xi_G)$, and consider the following natural estimator of $C_p''^G(u,v)$ defined as

$$\hat{C}_p''^G(u,v) = \frac{d^* - u\hat{A}_G^*}{3\sqrt{S_G^2 + v\hat{A}_G^2}},$$

where $\hat{A}_G = \max\{(1-\delta)^{-1}(\bar{\bar{G}} - T), (1+\delta)^{-1}(T - \bar{\bar{G}})\}$, $\hat{A}_G^* = (d^*/d)\hat{A}_G = (1-|\delta|)\hat{A}_G$, $\bar{\bar{G}} = \sum_{i=1}^r n_i\bar{G}_i/N$, and $S_G^2 = \sum_{i=1}^r n_i S_{G_i}^2/N$. Thus the cumulative distribution of $\hat{C}_p''^G(u,v)$ is

$$F_{\hat{C}_p''^G(u,v)}(x) = 1 - \int_0^{K_G(x)} H_G(x,t)dt, \quad \text{for} \quad x > 0, \quad (5)$$

where $K_G(x) = b_G\sqrt{N}/((1-|\delta|)u + 3x\sqrt{v})$, $b_G = 3C_p''^G$ with $C_p''^G = C_p''(0,0)$, $H_G(x,y) = F_K\left(\left(\frac{b_G\sqrt{N}-(1-|\delta|)ut}{3x}\right)^2 - vt^2\right)f_G(t)$, and $f_G(t) = (1-\delta)\phi((1-\delta)t - \sqrt{N}\xi_G) + (1+\delta)\phi((1+\delta)t + \sqrt{N}\xi_G)$.

4 Capability testing based on $\hat{C}_p''^{\mathsf{G}}(u,v)$

To determine whether a given process meets the preset capability requirement, we could consider the following statistical hypotheses testing:

– $H_0 : C_p''(u,v) \leqslant c$ Process is not capable;
– $H_1 : C_p''(u,v) > c$ Process is capable;

where c is the required process capability. If the calculated process capability $\hat{c}_p''(u,v)$ is greater than the critical value c_0, we reject the null hypothesis and conclude that the process is capable with error α, which is the chance of incorrectly concluding an incapable process as capable. Given values of c and α, from (4) the critical value c_0 can be determined by solving the equation $\alpha = P(\hat{C}_p''(u,v) > c_0|C_p''(u,v) = c) = \int_0^{K(c_0)} H(c_0,t)dt$, where from (A.6) in Appendix A, $b = 3C_p'' = 3\sqrt{1 + v\xi^{*2}}c + u(1-|\delta|)\xi^*$. However, since the process parameters μ and σ are unknown, parameters ξ and therefore ξ^* are unknown. For the particular cases C_{pk}, C_{pm}, and C_{pmk}, Pearn and Lin [26], Lin and Pearn [27], Hsu et al. [23], performed extensive calculations to obtain the maximum value of the critical value and showed that its maximum is obtained for the values of ξ equal respectively to 1, 0, and 0.5. Then they used those previous values in order to eliminate the necessity to estimate ξ. It is not possible to use the same method for any (u,v) pair, thus in the previous equations, we replace ξ and ξ^* by the observation of the natural estimators $\hat{\xi} = (\bar{\bar{X}} - T)/S$ and $\hat{\xi}^* = \max((1-\delta)^{-1}\hat{\xi}, -(1+\delta)^{-1}\hat{\xi})$. Note that we use the same symbols $\hat{\xi}$ and $\hat{\xi}^*$ for the estimators as well as the estimations.

In the presence of measurements errors, $C_p''^G(u,v)$ is measured rather than the true variable $C_p''(u,v)$. Thus the α-risk denoted by α_G is defined as $\alpha_G = P\left(\hat{C}_p''^G(u,v) > c_0|C_p''(u,v) = c\right)$. Since the process capability index is estimated by using $\hat{C}_p''^G(u,v)$ instead of $\hat{C}_p''(u,v)$, the real capability of the process is underestimated. The probability of $\hat{C}_p''^G(u,v)$ being greater than c_0 will be less important than by using $\hat{C}_p''(u,v)$. Thus α_G, the α-risk using $\hat{C}_p''^G(u,v)$ to estimate $C_p''(u,v)$ is less than α, using $\hat{C}_p''(u,v)$ when estimating $C_p''(u,v)$. To illustrate the performance of α_G versus λ, we have considered the particular case $u = 0.5$, $v = 1.2$, $\delta = 0.3$, $c = 1$ and 1.5, $\alpha = 0.05$, and have plotted curves α_G for various C_p'' values. The critical value c_0 is obtained from (4) where ξ is defined from (A.2) and ξ^* is defined from (A.15) as

See equation above.

Then from (5) we compute $\alpha_G = \int_0^{K_G(c_0)} H_G(c_0,t)dt$, where $C_p''^G$ and ξ_G are obtained from (A.10) and (A.12).

Figure 1 plots α_G versus $\lambda \in [0, 0.5]$ with $r = 1$, $N = 30$, 50, 70, 100, 150 for $C_p'' = c(0.5)(c+1)$. Note that for $\lambda = 0$, $\alpha_G = \alpha$. In Figure 1, α_G decreases as λ or N increases, and the decreasing rate is more significant with large c.

We now consider the power of the test, the chance of correctly concluding a capable process as capable. It can be computed as $\pi(C_p''(u,v)) = P(\hat{C}_p''(u,v) > c_0|C_p''(u,v)) = \int_0^{K(c_0)} H(c_0,t)dt$, with from (A.6), $b = 3C_p'' = 3\sqrt{1+v\xi^{*2}}C_p''(u,v) + u(1-|\delta|)\xi^*$.

In the presence of measurements errors, the power of the test denoted by π_G is as follows $\pi_G(C_p''(u,v)) = P(\hat{C}_p''^G(u,v) > c_0|C_p''(u,v)) = \int_0^{K_G(c_0)} H_G(c_0,t)dt$. To illustrate the performance of π_G versus λ, we consider the particular case $u = 0.5$, $v = 1.2$, $\delta = 0.3$, and plot the curves π_G for various C_p'' and $C_p''(u,v)$ values. In the previous equation, $C_p''^G$, ξ^*, ξ and ξ_G are obtained from (A.10), (A.15), (A.2) and (A.12). Figure 2 plots π_G versus $\lambda \in [0, 0.5]$ with $r = 1$, $N = 50$, $\alpha = 0.05$, $c = 1.00$ and 1.50, $C_p''(u,v) = c(0.20)(c + 1)$, and $C_p'' = C_p''(u,v)$, $C_p''(u,v) + 0.33$ and $C_p''(u,v) + 0.5$. In Figure 2, we see that π_G decreases as λ increases and the decreasing rate is more significant with large c. The presence of measurements errors can have a very substantial effect on π_G. For instance, for $c = 1.5$ and $C_p''(u,v) = 2.5$, π_G is approximately equal to 1 without measurement errors and approximately equal to 0 when $\lambda = 0.5$.

5 Adjusted critical values

As we have seen in the previous section, the α-risk and the test power decrease with measurement errors. The capability testing results would be misleading if the producers do not take account of the gauge measurements errors. Thus, in order to improve the test power, we revise the critical value, denoted by c_0^A, to satisfy $c_0^A < c_0$. Let $\alpha_A = P(\hat{C}_p''^G(u,v) > c_0^A|C_p''(u,v) = c)$ and $\pi_A(C_p''(u,v)) = P(\hat{C}_p''^G(u,v) > c_0^A|C_p''(u,v))$, be the α-risk and the test power using the adjusted critical value. Since $c_0^A < c_0$, $P(\hat{C}_p''^G(u,v) > c_0^A)$ is greater than $P(\hat{C}_p''^G(u,v) > c_0)$, and both π_A and α_A increase. To ensure that the α-risk is within the preset magnitude, we set $\alpha_A = \alpha$, and solve equation $\alpha = P(\hat{C}_p''^G(u,v) > c_0^A|C_p''(u,v) = c)$ to obtain c_0^A. Thus from (5), we must solve

$$\alpha = \int_0^{K_G(c_0^A)} H_G(c_0^A,t)dt.$$

Since the process parameters μ and σ_G are unknown, then ξ_G is also unknown. Thus the previous equation involves the unknown parameters ξ_G and $C_p''^G$ or C_p'' from (A.10). We replace ξ_G by the observed value $\hat{\xi}_G$, and from (A.11),

C_p'' can be obtained by solving equation

$$\frac{C_p''}{\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2}} = \frac{\sqrt{1+v\hat{\xi}_G^{*2}}\sqrt{1+v\hat{\xi}^{*2}}}{\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2 + v\hat{\xi}^{*2}}} \times c + u(1-|\delta|)\hat{\xi}_G^*/3, \tag{6}$$

where $\hat{\xi}_G^*$ and $\hat{\xi}^*$ are defined from (A.3) and (A.14) as follows

$$\hat{\xi}_G^* = \max((1-\delta)^{-1}\hat{\xi}_G, -(1+\delta)^{-1}\hat{\xi}_G), \tag{7}$$

and

$$\hat{\xi}^* = \max\left(\frac{\hat{\xi}_G}{1-\delta}\sqrt{1+\lambda^2\left(\frac{\sqrt{1+v\hat{\xi}^{*2}}}{1-|\delta|}c + u\hat{\xi}^*/3\right)^2}, \right.$$
$$\left. -\frac{\hat{\xi}_G}{1+\delta}\sqrt{1+\lambda^2\left(\frac{\sqrt{1+v\hat{\xi}^{*2}}}{1-|\delta|}c + u\hat{\xi}^*/3\right)^2}\right). \tag{8}$$

With the adjusted critical value c_0^A, we can calculate the test power noted down as π_A. Let C_{p1}'' be the value of C_p'' the solution of equation (6). The test power can be calculated as follows
$\pi_A(C_p''(u,v)) = P\left(\hat{C}_p''^G(u,v) > c_0^A|C_p''(u,v), C_p'' = C_{p1}''\right) = \int_0^{K_1(c_0^A)} H_1(c_0^A,t)dt$, with $K_1(x) = b_1\sqrt{N}/((1-|\delta|)u + 3x\sqrt{v})$,
$b_1 = 3C_{p1}''^G = \dfrac{3C_{p1}''}{\sqrt{1+\lambda^2 C_{p1}''^2/(1-|\delta|)^2}}$ from (3),

$$H_1(x,y) = F_K\left(\left(\frac{b_1\sqrt{N}-(1-|\delta|)ut}{3x}\right)^2 - vt^2\right)f_{G1}(t),$$

$f_{G1}(t) = (1-\delta)\phi\left((1-\delta)t - \sqrt{N}\xi_{G1}\right) + (1+\delta)\phi((1+\delta)t + \sqrt{N}\xi_{G1})$, and

$$\xi_{G1} = \begin{cases} (1-\delta)\xi_{G1}^* & \text{if } \xi_{G1} > 0 \\ -(1+\delta)\xi_{G1}^* & \text{if } \xi_{G1} < 0 \end{cases} \text{ from (A.4).}$$

ξ_1^* then ξ_{G1}^* are obtained by substituting C_{p1}'' to C_p'' in (A.15) and (A.13). ξ_{G1} and ξ_G have the same sign, but ξ_G is unknown. Thus we replace ξ_G by the observed value $\hat{\xi}_G$, and replace ξ_{G1} by $\hat{\xi}_{G1}$ defined by

$$\hat{\xi}_{G1} = \begin{cases} (1-\delta)\xi_{G1}^* & \text{if } \hat{\xi}_G > 0 \\ -(1+\delta)\xi_{G1}^* & \text{if } \hat{\xi}_G < 0. \end{cases}$$

A Maple program is developed in Appendix B to compute the adjusted critical value c_0^A and plot the power test.
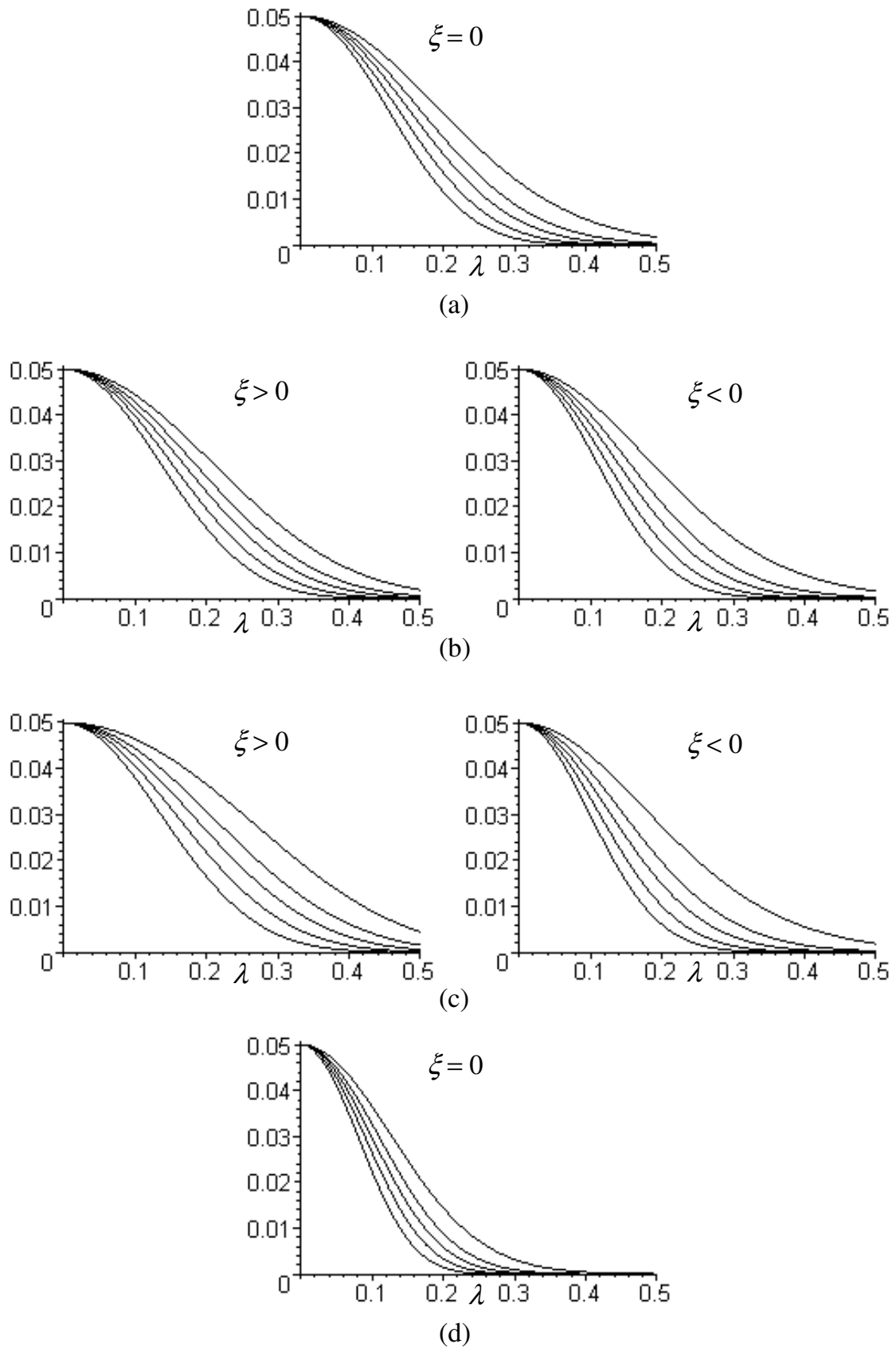
Fig. 1. Plots of α_G versus λ with $r = 1$, $N = 30, 50, 70, 100, 150$ (top to bottom), $u = 0.5$, $v = 1.2$, $\delta = 0.3$, $\alpha = 0.05$ for (a) $c = 1$, $C_p'' = 1$; (b) $c = 1$, $C_p'' = 1.5$; (c) $c = 1$, $C_p'' = 2$; (d) $c = 1.5$, $C_p'' = 1.5$; (e) $c = 1.5$, $C_p'' = 2$; (f) $c = 1.5$, $C_p'' = 2.5$.
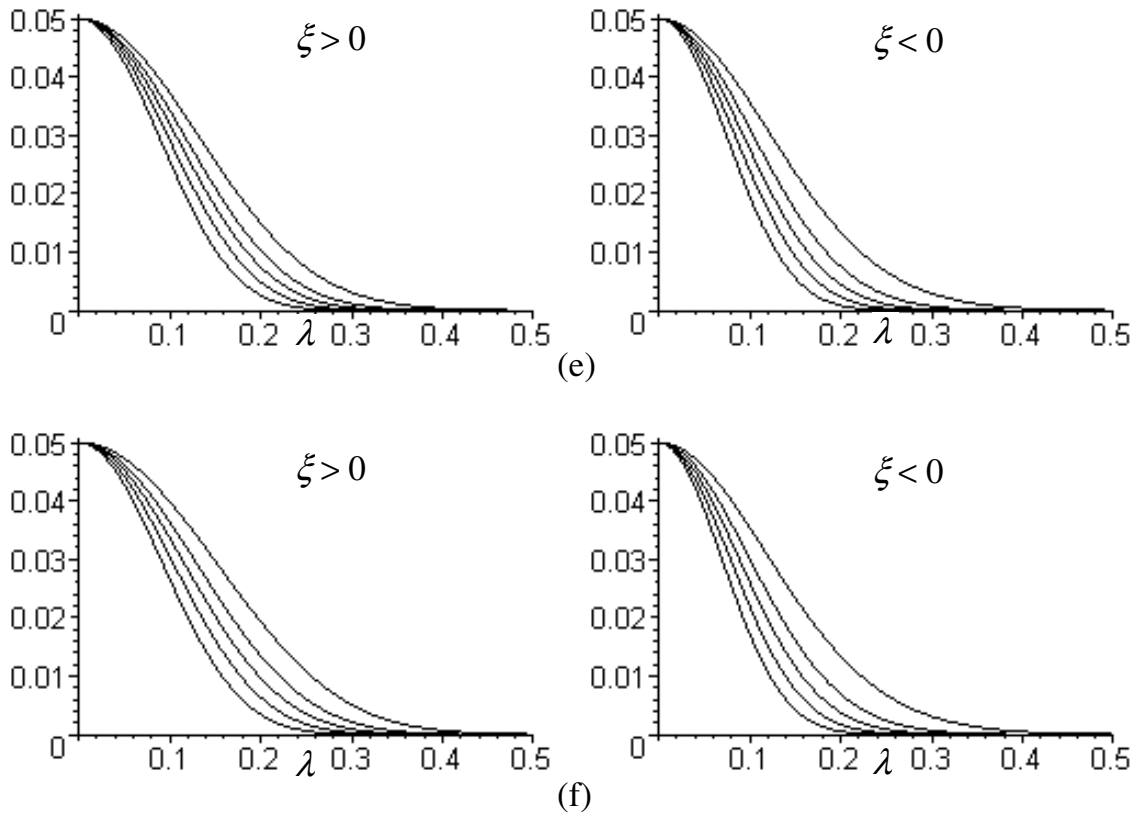
Fig. 1. Continued.

Figure 3 plots π_A versus $\lambda \in [0, 0.5]$ with $r = 1$, $N = 50$, $\alpha = 0.05$, $c = 1.00$, 1.50, $C_p''(u, v) = c(0.20)(c + 1)$ and $C_p'' = C_p''(u, v)$, $C_p''(u, v)+0.5$. In Figure 3, π_A decreases as increases and the decreasing rate is more significant with large c. However, if we compare Figures 2 and 3, we can see that the powers corresponding to the adjusted critical values c_0^A decrease more slowly. Thus the test power is improved.
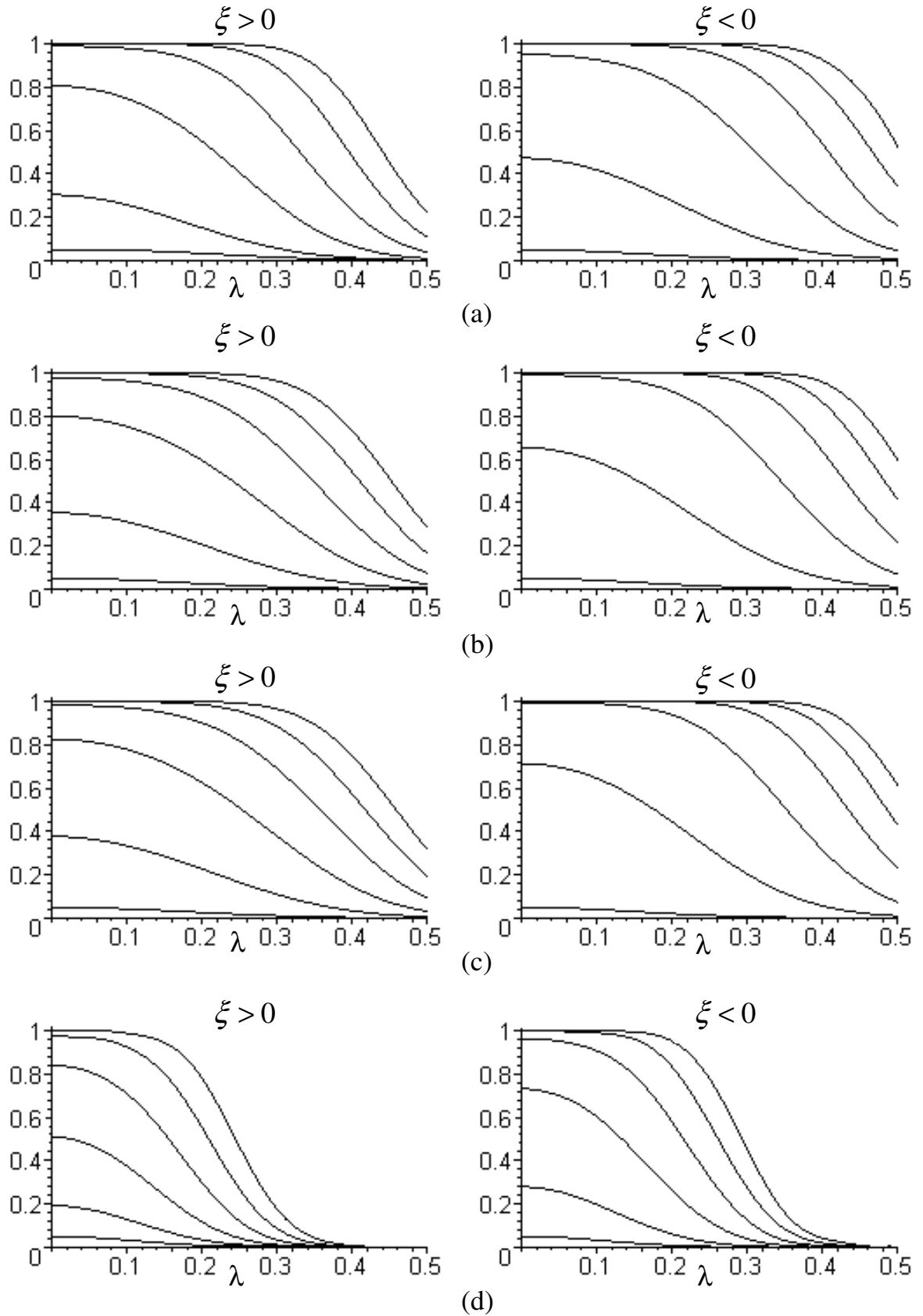
6 Example

The study we present here has been conducted within the company Chabert and Guillot producing nougat bars made with almonds, honey and other natural ingredients. At the output of the production, the nougat comes in the form of a 520 mm long block. A saw cuts this block into 40 of 13 mm bars, a weight sorter ejects the too heavy or too light bars, which are then recycled. Sets of about 10 000 bars are then packaged. Bars weighing 200 g are sold. For a batch to be accepted, the French legislation requires that no bar should weigh less than 182 g, there should not be more than 2% of bars between 182 g and 191 g, and the mean weight should be at least 200 g. Given the legislative and financial regulations, target T has been set at 212 g and L and U tolerances at 191 g and 230 g (values beyond which the bars are recycled). We have $d^* = 18$, and $\delta = 0.077$. To determine the process

capability, 20 blocks have been randomly selected. Due to a systematic deformation at the beginning and the end of the block, only 36 bars are used to determine the mean weight and the standard deviation of the weight of a bar. The data collected are presented in Table 1. Histogram plot shows that the collected data follow the normal distribution and the Shapiro-Wilk test is applied to further justify the assumption.

To choose an index $C_p''(u, v)$, Grau [11] suggests that it should be defined so that the process cannot be claimed as capable when the mean deviation and the proportion of non-conforming exceed the maximum limits. On the other hand, in order to have an easily interpretable measure, the process is said capable when $C_p''(u, v)$ is higher than or equal to 1. For the firm, the process cannot be considered capable if the mean is less than 200 g, that is to say if the mean moves away from more than $K = (T - \mu)D_l = 57\%$ of the distance between the target and the lower limit. Using the results Grau [11] obtained, for u and v values defined with a precision of 0.1, Table 2 lists the pairs which get as close as possible to this requirement. This table also shows the maximum proportion M of non-conforming bars when $C_p''(u, v) = 1$. Given the annual costs of excessive dosage, the company has decided to use the $C_p''(0.8, 0.1)$ index.

To determine whether the process is capable with a gauge capability λ equal to 0.12 provided by a R&R

Fig. 2. Plots of π_G versus λ with $u = 0.5$, $v = 1.2$, $\delta = 0.3$, $r = 1$, $N = 50$, $\alpha = 0.05$, $C_p''(u,v) = c\,(0.20)(c+1)$ (bottom to top) for (a) $c = 1$ and $C_p'' = C_p''(u,v)$; (b) $c = 1$ and $C_p'' = C_p''(u,v) + 0.33$; (c) $c = 1$ and $C_p'' = C_p''(u,v) + 0.5$; (d) $c = 1.5$ and $C_p'' = C_p''(u,v)$; (e) $c = 1.5$ and $C_p'' = C_p''(u,v) + 0.33$; (f) $c = 1.5$ and $C_p'' = C_p''(u,v) + 0.5$.

(e)

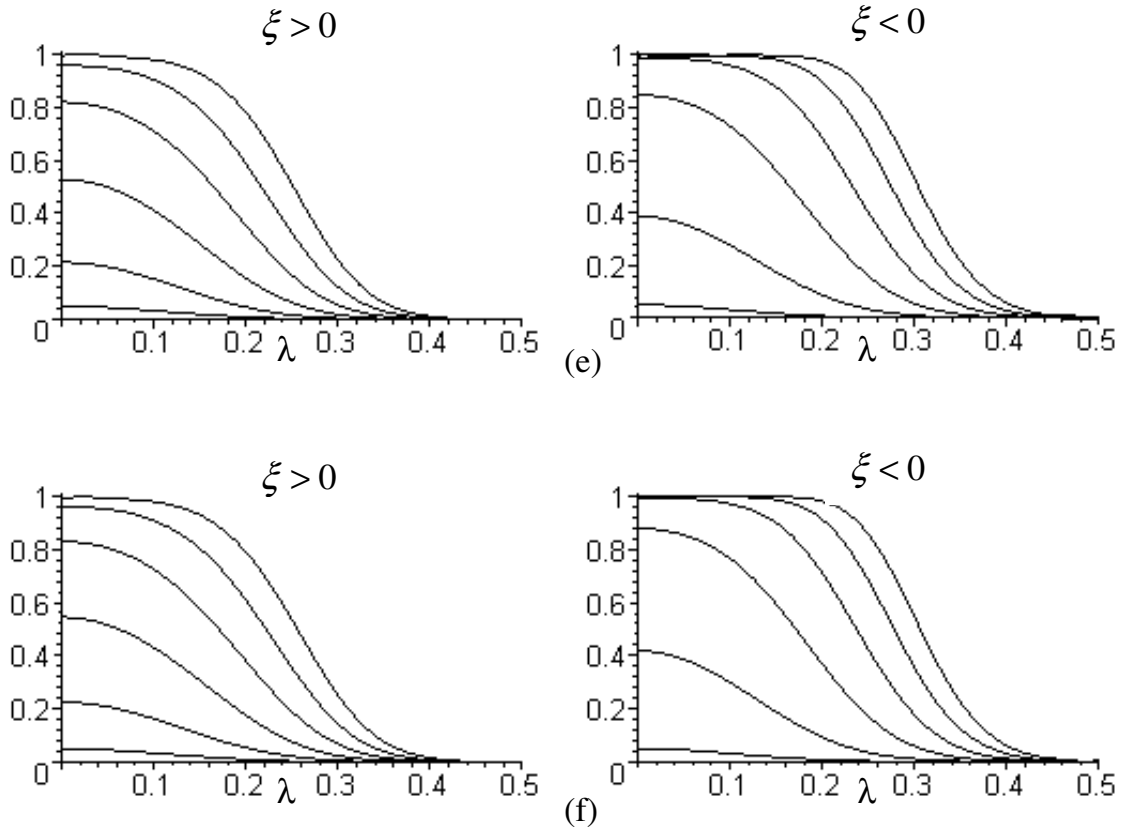
(f)

Fig. 2. Continued.

Table 1. Means and standard deviations of the 20 subsamples.

Sample number	1	2	3	4	5	6	7
$\bar{x}_i$	208.506	210.116	208.797	209.814	210.463	209.562	209.897
s_i	4.712	3.648	3.264	3.437	4.418	4.518	4.225
Sample number	8	9	10	11	12	13	14
$\bar{x}_i$	209.553	209.286	209.577	210.851	210.168	211.011	211.018
s_i	4.632	4.825	4.654	4.365	4.153	4.419	5.170
Sample number	15	16	17	18	19	20	
$\bar{x}_i$	209.464	210.554	209.582	209.776	211.002	210.882	
s_i	4.709	4.695	4.258	4.943	4.834	3.946	

Table 2. Pairs (u, v) for which K is roughly equal to 57% when $C_p''(u, v) = 1$, and M the maximum percentage of non-conforming.

| (u, v)	(0, 0.3)	(0.1, 0.3)	(0.3, 0.2)	(0.4, 0.2)	(0.7, 0.1)	(0.8, 0.1)
K	56%	53%	57%	54%	58%	55%
M in ppm	6532	4821	6127	3143	2057	1628

analysis, we determine that $c = 1$ and $\alpha = 0.05$. Then, based on the sample data of $N = 720$ observations, we obtain $\bar{\bar{G}} = 209.994$, $S_G = 4.418$, $\hat{\xi}_G = -0.454$ and $\hat{c}_p''^G(0.8, 0.1) = 1.282$. From the Maple program in Appendix B, we obtain the critical value $c_0^A = 1.052$. Since $\hat{c}_p''^G(0.5, 0.1) > c_0^A$, we can conclude that the process is capable. We also see that if we ignore the measurements errors, the critical value may be calculated as $c_0 = 1.063$. So if another sample gives $\bar{\bar{G}} = 209.590$ and $S_G = 5.307$, we

always have $\hat{\xi}_G = -0.454$, thus $c_0^A = 1.052$ and $c_0 = 1.063$. Since $\hat{c}_p''^G(0.8, 0.1) = 1.056$, we conclude that the process is capable, but we cannot draw the same conclusion if we ignore the measurement errors.

7 Conclusion

In this work we have been concerned with the problem of gauge measurement errors when dealing with process
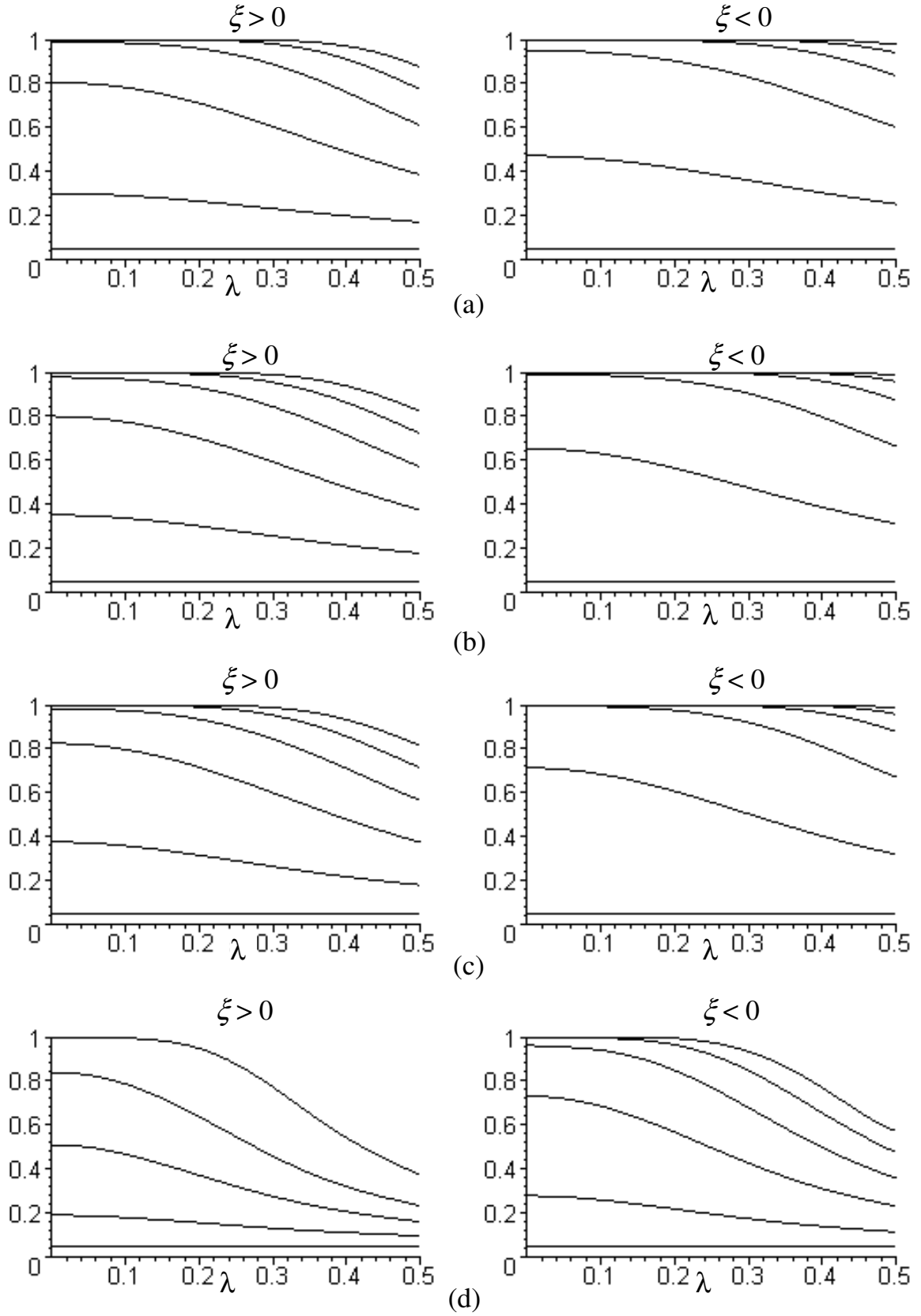
Fig. 3. Plots of π_A versus with $u = 0.5$, $v = 1.2$, $\delta = 0.3$, $r = 1$, $N = 50$, $\alpha = 0.05$, $C_p''(u, v) = c \, (0.20)(c + 1)$ (bottom to top) for (a) $c = 1$ and $C_p'' = C_p''(u, v)$; (b) $c = 1$ and $C_p'' = C_p''(u, v) + 0.33$; (c) $c = 1$ and $C_p'' = C_p''(u, v) + 0.5$; (d) $c = 1.5$ and $C_p'' = C_p''(u, v)$; (e) $c = 1.5$ and $C_p'' = C_p''(u, v) + 0.33$; (f) $c = 1.5$ and $C_p'' = C_p''(u, v) + 0.5$.
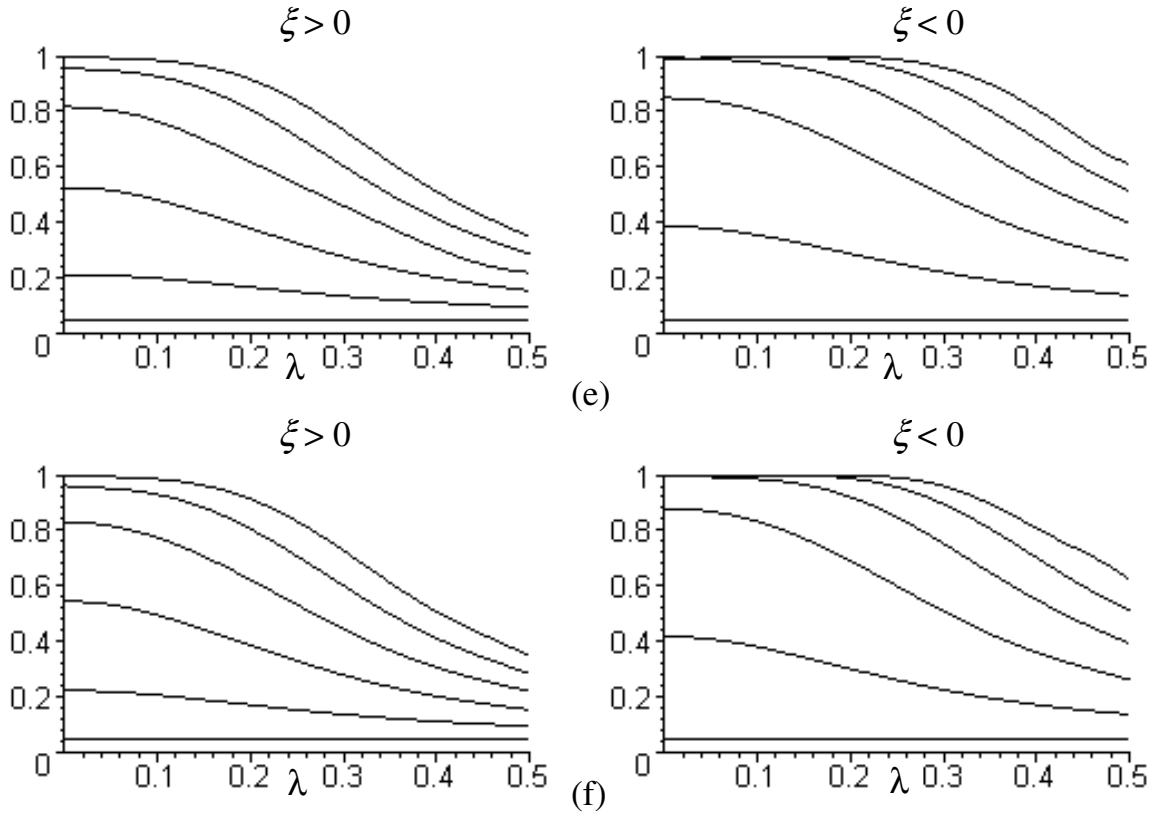
Fig. 3. Continued.

capability indices for symmetric or asymmetric tolerances. Gauge measurement errors have a significant impact on the estimation of a process capability. When small sub-samples data are collected from past "in-control", the probability distribution of the estimators of the process capability $C_p''(u, v)$ and the process capability with gauge measurement errors $C_p''^G(u, v)$ has been investigated to obtain critical values and power test. The estimator evaluating the capability by using the sample data contaminated with gauge measurement errors severely underestimates the α-risk, resulting in a smaller testing power. We suggest an adjusted critical value to improve the accuracy of the capability assessment. For practical purpose we give a Maple program helpful to the users for their factory applications. Finally a real example in a company producing nougat is used to assess the interest of the approach.

Appendix A

– We have

$$\xi^* = \max((1 - \delta)^{-1}\xi, -(1 + \delta)^{-1}\xi) \tag{A.1}$$

$$\xi = \begin{cases} (1 - \delta)\xi^* & \text{if } \xi > 0 \\ -(1 + \delta)\xi^* & \text{if } \xi < 0 \end{cases} \tag{A.2}$$

$$\xi_G^* = \max((1 - \delta)^{-1}\xi_G, -(1 + \delta)^{-1}\xi_G) \tag{A.3}$$

$$\xi_G = \begin{cases} (1 - \delta)\xi_G^* & \text{if } \xi_G > 0 \\ -(1 + \delta)\xi_G^* & \text{if } \xi_G < 0 \end{cases} \tag{A.4}$$

– Since $C_p = C_p(0, 0)$ and $C_p'' = C_p''(0, 0)$, we have

$$C_p = C_p''/(1 - |\delta|). \tag{A.5}$$

– Since $\xi^* = A/\sigma$ and $A^* = (1 - |\delta|)A$, from (1) we have $C_p''(u, v) = \frac{C_p'' - uA^*/(3\sigma)}{\sqrt{1 + vA^2/\sigma^2}} = \frac{C_p'' - u(1 - |\delta|)\xi^*/3}{\sqrt{1 + v\xi^{*2}}}$, thus

$$C_p'' = \sqrt{1 + v\xi^{*2}} C_p''(u, v) + u(1 - |\delta|)\xi^*/3. \tag{A.6}$$

– In the same manner, from (2) we have

$$C_p''^G = \sqrt{1 + v\xi_G^{*2}} C_p''^G(u, v) + u(1 - |\delta|)\xi_G^*/3. \tag{A.7}$$

– From (A.5) and (A.6) we have

$$C_p = \sqrt{1 + v\xi^{*2}} C_p''(u, v)/(1 - |\delta|) + u\xi^*/3. \tag{A.8}$$

– From (A.7) and (3), we have

$$C_p''^G = \frac{\sqrt{1 + v\xi_G^{*2}} \sqrt{1 + v\xi^{*2}}}{\sqrt{1 + \lambda^2 C_p''^2/(1 - |\delta|)^2 + v\xi^{*2}}}$$
$$\times C_p''(u, v) + u(1 - |\delta|)\xi_G^*/3. \tag{A.9}$$

– Since $C_p''^G = C_p''^G(0, 0)$, from (3) we have

$$C_p''^G = C_p''/\sqrt{1 + \lambda^2 C_p''^2/(1 - |\delta|)^2}. \tag{A.10}$$

$$\xi^* = \frac{-C_0 u(1-|\delta|)/3 \pm \sqrt{C^2\left(v\left(C_0^2 - C^2\right) + (u(1-|\delta|)/3)^2\right)}}{vC^2 - (u(1-|\delta|)/3)^2}$$

$$= \frac{-C_0 u(1-|\delta|)/3 \pm \sqrt{C_0^2\left(u(1-|\delta|)/3\right)^2 + \left(C_0^2 - C^2\right)\left(vC^2 - (u(1-|\delta|)/3)^2\right)}}{vC^2 - (u(1-|\delta|)/3)^2}$$

$$= \frac{-a \pm \sqrt{a^2 + \alpha b}}{b},$$

– From (A.10) and (A.9), we have

$$\frac{C_p''}{\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2}} =$$

$$\frac{\sqrt{1+v\xi_G^{*2}}\,\sqrt{1+v\xi^{*2}}}{\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2 + v\xi^{*2}}} C_p''(u,v)$$

$$+ u(1-|\delta|)\xi_G^*/3. \quad \text{(A.11)}$$

– Since $\xi = (\mu - T)/\sigma$ and $\xi_G = (\mu - T)/\sigma_G$, we have $\xi_G = \xi\sigma/\sigma_G = \xi\sigma/\sqrt{\sigma^2 + \sigma_M^2} = \xi/\sqrt{1+\sigma_M^2/\sigma^2} = \xi/\sqrt{1+\lambda^2 C_p^2}$, and from (A.5) we have

$$\xi_G = \xi/\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2}. \quad \text{(A.12)}$$

– From (A.3), (A.12) and (A.1), we have

$$\xi_G^* = \xi^*/\sqrt{1+\lambda^2 C_p''^2/(1-|\delta|)^2}. \quad \text{(A.13)}$$

– From (A.1), (A.12), (A.5) and (A.8), we have

$$\xi^* = \max\left(\frac{\xi_G}{1-\delta}\sqrt{1+\lambda^2\left(\frac{\sqrt{1+v\xi^{*2}}}{1-|\delta|}C_p''(u,v)+u\xi^*/3\right)^2},\right.$$

$$\left.-\frac{\xi_G}{1+\delta}\sqrt{1+\lambda^2\left(\frac{\sqrt{1+v\xi^{*2}}}{1-|\delta|}C_p''(u,v)+u\xi^*/3\right)^2}\right). \quad \text{(A.14)}$$

– We have $C_p''(u,v) = \frac{d^*-uA^*}{3\sqrt{\sigma^2+vA^2}} = \frac{C_p''-uA^*/(3\sigma)}{\sqrt{1+vA^2/\sigma^2}} = \frac{C_p''-u(1-|\delta|)\xi^*/3}{\sqrt{1+v\xi^{*2}}}$

$$\Leftrightarrow \begin{cases} \left(C_p''(u,v)\right)^2(1+v\xi^{*2}) = \left(C_p''-u\xi^*(1-|\delta|)/3\right)^2 & \text{(a)} \\ \xi^* < 3C_p''/(u(1-|\delta|)) & \text{(b)} \end{cases}$$

Assume that $C = C_p''(u,v)$ and $C_0 = C_p''$.

$$(a) \Leftrightarrow C^2 + vC^2\xi^{*2} = C_0^2 + \xi^{*2}\left(u(1-|\delta|)/3\right)^2$$
$$- 2C_0 u\xi^*(1-|\delta|)/3$$
$$\Leftrightarrow \left(vC^2 - (u(1-|\delta|)/3)^2\right)\xi^{*2} + 2C_0 u\xi^*(1-|\delta|)/3$$
$$+ C^2 - C_0^2 = 0,$$

which is a second-degree polynomial of the variable ξ^*, and the discriminant is

$$\Delta' = (C_0 u(1-|\delta|)/3)^2 - (vC^2 - (u(1-|\delta|)/3)^2)$$
$$\times (C^2 - C_0^2)$$
$$= \left(vC^2\left(C_0^2 - C^2\right) + C^2\left(u(1-|\delta|)/3\right)^2\right)$$
$$= C^2\left(v\left(C_0^2 - C^2\right) + (u(1-|\delta|)/3)^2\right).$$

Then $\alpha = C_0^2 - C^2 = \left(C_p''\right)^2 - \left(C_p''(u,v)\right)^2 \geqslant 0$, thus $\Delta' \geqslant 0$, and

See Equation above.

with $a = C_0 u(1-|\delta|)/3 \geqslant 0$ and $b = vC^2 - (u(1-|\delta|)/3)^2$.

Thus the solutions of the equation are $\xi_1^* = \frac{-a+\sqrt{a^2+\alpha b}}{b}$ and $\xi_2^* = \frac{-a-\sqrt{a^2+\alpha b}}{b}$. However $\xi^* = \max((1-\delta)^{-1}\xi, -(1+\delta)^{-1}\xi)$ must necessarily be positive or equal to zero.

(1) If $C = C_p''(u,v) > \frac{u(1-|\delta|)}{3\sqrt{v}}$, then $b > 0$ and $\xi_2^* < 0$ is not suitable.

(2) If $0 < C = C_p''(u,v) < \frac{u(1-|\delta|)}{3\sqrt{v}}$, then

$$0 < vC^2 < (u(1-|\delta|)/3)^2 \Leftrightarrow -(u(1-|\delta|)/3)^2 < b < 0.$$
Thus
$$a^2 - \alpha\left(u(1-|\delta|)/3\right)^2 < a^2 + \alpha b$$

$$\Leftrightarrow C_0^2(u(1-|\delta|)/3)^2 - (C_0^2 - C^2)(u(1-|\delta|)/3)^2 < a^2 + \alpha b$$

$$\Leftrightarrow C^2(u(1-|\delta|)/3)^2 < a^2 + \alpha b \Leftrightarrow C(u(1-|\delta|)/3) < \sqrt{a^2+\alpha b} \Leftrightarrow \frac{-a-Cu(1-|\delta|)/3}{b} < \frac{-a-\sqrt{a^2+\alpha b}}{b}, \text{ since } b < 0$$

$$\Leftrightarrow \frac{-a - Cu(1-|\delta|)/3}{vC^2 - (u(1-|\delta|)/3)^2} < \xi_2^* \Leftrightarrow \frac{a + Cu(1-|\delta|)/3}{(u(1-|\delta|)/3)^2 - vC^2} < \xi_2^*$$

$$\Leftrightarrow \frac{a+Cu(1-|\delta|)/3}{(u(1-|\delta|)/3)^2} < \xi_2^* \Leftrightarrow \frac{a}{(u(1-|\delta|)/3)^2} < \xi_2^*$$

$$\Leftrightarrow \frac{3}{u(1-|\delta|)}C_0 < \xi_2^*$$

$\Leftrightarrow \xi_2^* > 3C_p''^G/(u(1-|\delta|))$, is a solution that does not check (b), thus it is not suitable. Therefore the only possible solution is $\xi^* = \xi_1^* = \frac{-a+\sqrt{a^2+\alpha b}}{b}$, that is to say

$$\xi^* = \frac{-C_p'' u(1-|\delta|)/3 + \sqrt{\left(C_p''\right)^2 (u(1-|\delta|)/3)^2 + \left(\left(C_p''\right)^2 - \left(C_p''(u,v)\right)^2\right)\left(v\left(C_p''(u,v)\right)^2 - (u(1-|\delta|)/3)^2\right)}}{v\left(C_p''(u,v)\right)^2 - (u(1-|\delta|)/3)^2} \tag{A.15}$$

See equation (A.15) above.

Appendix B

When the function H_G takes values that are too low, the Maple software is sometimes unable to perform the calculation of the integral of H_G. This is the reason why we limit the domain of integration $[0; K_G]$ to the domain $[K0; K1]$ on which $H_G > 10^{-8}$. This does not change the value of the integral for the required precision.

Maple Program to compute the adjusted critical value c_0^A and the power test.

Algorithm

Step 1. Read $u, v, \delta, \lambda, r, N, \alpha, c, \hat{\xi}_G$
Step 2. Compute $\hat{\xi}^*$ from (8), $\hat{\xi}_G^*$ from (7), C_p'' from (6), $C_p''^G$ from (A.10).
Step 3. Compute H_G, K_G.
Step 4. Find the adjusted critical value c_0^A.
Step 4.1. Find $K0$ a lower integration bound.
Step 4.2. Find $K1$ an upper integration bound.
Step 5. Print the adjusted critical value.
Step 6: Plot the power test.

Maple program
restart:with(stats):

#1. Read u, v, delta, lambda, r (sub-samples number), N (total number of observations), alpha (the alpha-risk), c (threshold value for a capable process), kG (estimation of (muG-T)/sigmaG), cpuvG (estimation of C″ puvG, to obtain the power test)
u:=0.8:v:=0.1:delta:=0.077:lambda:=0.12:
r:=20:N:=720:alpha:=0.05:c:=1:kG:=-0.454:

#2. Compute ks (estimation of xi_star), kGs (estimation of xiG_star), Cp (C″p), CpG (C″pG)
ks:=solve(x=piecewise(kG>0,kG/(1-delta)*sqrt(1+lambda^2*(sqrt(1+v*x^2)/(1-abs(delta))*c+u*x/3)^2),-kG/(1+delta)*sqrt(1+lambda^2*(sqrt(1+v*x^2)/(1-abs(delta))*c+u*x/3)^2)),x):
kGs:=max(kG/(1-delta),-kG/(1+delta)):
Cp:=fsolve(x/sqrt(1+lambda^2*x^2/(1-abs(delta))^2)=sqrt(1+v*kGs^2)*sqrt(1+v*ks^2)*c/sqrt(1+lambda^2*x^2/(1-abs(delta))^2+v*ks^2)+u*(1-abs(delta))*kGs/3,x=0..10):
CpG:=Cp/sqrt(1+lambda^2*Cp^2/(1-abs(delta))^2):

#3. Compute H (HG) and K (KG)
F:=(c0,t)->stats[statevalf,cdf,chisquare[N-r]]((3*CpG*sqrt(N)-(1-abs(delta))*u*t)^2/9/c0^2-v*t^2):

f:=t->(1-delta)*stats[statevalf,pdf,normald]((1-delta)*t-kG*N^.5)+(1+delta)*stats[statevalf,pdf,normald]((1+delta)*t+kG*N^.5):
H:=(c0,t)->F(c0,t)*f(t):
K:=3*CpG*sqrt(N)/((1-abs(delta))*u+3*c0*sqrt(v)):

#4. Compute c0 (c0A) the adjusted critical value
k1:=0:k3:=5.9:
for i from 1 to 100
do k2:=(k1+k3)/2:c0:=evalf(k2):
#4.1. Find K0, lower integration bound
    K0:=0:z1:=evalf(H(c0,K0)):
    if z1<1e-8 then for j from 1 to trunc(K)
    do z1:=evalf(H(c0,j));
    if z1>1e-8 then for h from j-0.9 by 0.1 to j
    do z1:=evalf(H(c0,h));
    if z1>1e-8 then K0:=h-0.1:break fi:
    od:
    break fi:
    od:
    fi:
#4.2. Find K1, upper integration bound
K1:=evalf(K)-0.00001: z2:=evalf(H(c0,K1)):
if z2<1e-8 then for j from K1 by -1 to 0
do z2:=evalf(H(c0,j));
if z2>1e-8 then for h from j+0.9 by -0.1 to j
    do z2:=evalf(H(c0,h));
    if z2>1e-8 then K1:=h+0.1:break fi:
    od:
break fi:
od:
fi:
if z1<1e-8 and z2<1e-8 then y:=0
else y:=evalf(int(H(c0,t),
    t=evalf(K0)..evalf((K0+K1)/2)))
    +evalf(int(H(c0,t),t=evalf((K0+K1)/2)..evalf(K1))):
    fi:
    if y<alpha then k3:=k2 else k1:=k2 fi:
    if abs(y-alpha)<0.00001 then break fi:
    od:

#5. Print the adjusted critical value.
    printf("the adjusted critical value is %.3f with %.2f per cent risk",c0,100*alpha);

#6. Plot the power test
ks1:=(-Cp*u*(1-abs(delta))/3+sqrt(Cp^2*(u*(1-abs(delta))/3)^2+(Cp^2-Cpuv^2)*(v*Cpuv^2-(u*(1-abs(delta))/3)^2)))/(v*Cpuv^2-(u*(1-abs(delta))/3)^2):
kGs1:=ks1/sqrt(1+lambda^2*Cp^2/(1-abs(delta))^2):
kG1:=piecewise(kG>0,(1-delta)*kGs1,-(1+delta)*kGs1):

```
f:=t->(1-delta)*stats[statevalf,pdf,normald]((1-delta)*t-
kG1*N^.5)+(1+delta)*stats[statevalf,pdf,normald]
((1+delta)*t+kG1*N^.5):
pw:=c0->int(F(c0,t)*f(t),t=0..K):
plot(pw(c0),Cpuv=c..Cp-0.01,title="test power");
```

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